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The Potential of a Lens, and Allied Physical Problems.

By G. Greenhill.

As a sequel to the article in this Journal, Vol. XXXIII, p. 311, on the "Potential of a Spherical Segment (a Lens)", some further applications may be cited, by which the solution can be given of allied problems in Electricity, Attraction, and Hydrodynamics.

1. Starting with U, the potential function (P. F.) of a spherical bowl of radius c cm, and uniform superficial density, σ, in g/cm^2 super, it was shown in the former article (cited as A. J. M.) that G denoting the gravitation constant, 666×10^{-10} in C. G. S. units,

$$\frac{U}{G\sigma} = c\Omega + r'\Omega', \tag{1}$$

(A. J. M., § 8), and this is Maxwell's P of § 670, "Electricity and Magnetism" (E. and M.); and then, as in A. J. M., § 13 (18),

$$\frac{1}{c}\frac{d}{dr}(Pr) = \frac{d}{dr}(r\Omega + c\Omega') = \Omega + r\frac{d\Omega}{dr} + c\frac{d\Omega'}{dr} = \Omega. \tag{2}$$

The P. F. U in (1) is composed of two terms, of which Ω in the first is the apparent area at P of the bowl or its base AB (fig. 1), and Ω is a P. F. satisfying Laplace's equation; and as U is a P. F. at P, it follows that the second term $r'\Omega'$ is also a P. F. at P, while Ω' is a P. F. at P', the inverse point of P in the spherical surface, Ω' representing the apparent area of the bowl or its base AB at P'.

Interpreted physically, Ω will represent the magnetic potential at P of a plate bounded by the circle AB and magnetized normally, or the equivalent electro-magnetic potential of a current round the rim; or it will represent the illumination on a page at P parallel to AB, due to skylight coming through the circle AB, as inside a chimney shaft or down a well, where the illumination of a sky of uniform brightness would be reduced by the fraction

$$\frac{\Omega}{2\pi} = 1 - \frac{h}{\sqrt{(h^2+a^2)}},$$

at a depth h in the middle of a shaft or well of radius a.

2. The radial component F of the attraction along PC is given by

$$\frac{F}{G\sigma}=-\frac{d}{dr}(c\Omega+r'\Omega')=-c\frac{d\Omega}{dr}-r'\frac{d\Omega'}{dr}+\frac{c^2}{r^2}\Omega'=\frac{c^2}{r^2}\Omega'. \tag{1}$$

Thus with P on the axis, and outside the convex side of the bowl at G, when P' is inside at O,

$$\Omega'=2\pi,\quad \frac{F}{G\sigma}=2\pi\frac{CA^2}{CG^2}=2\pi\cos^2\gamma. \tag{2}$$

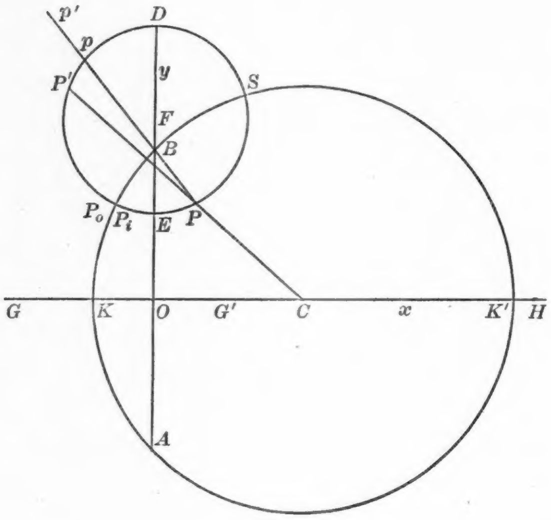

FIG. 1.

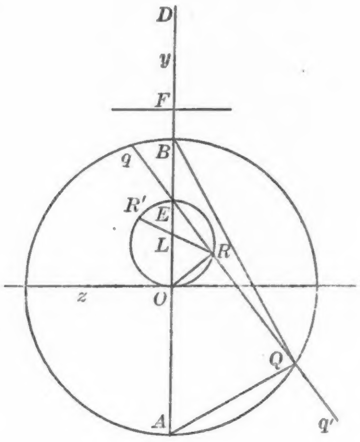

FIG. 2.

With P inside at O, and P' outside at G, Ω' is negative,

$$\Omega'=-4\pi+2\pi\frac{OG}{OA}=-4\pi+2\pi\sin\gamma,$$

$$\frac{F}{G\sigma}=2\pi(-2+\sin\gamma)\frac{CA^2}{CO^2}=2\pi\frac{-2+\sin\gamma}{\cos^2\gamma}. \tag{3}$$

With P at K', $c=r=x'$, $\Omega=\Omega'$,

$$\frac{F}{G\sigma}=\Omega=2\pi\frac{OK'}{AK'}=2\pi\cos\tfrac{1}{2}\gamma; \tag{4}$$

P outside, close to K, and P' inside, $r=c=r'$,

$$\frac{F}{G\sigma}=\Omega'=2\pi\frac{KO}{KA}=2\pi\sin\tfrac{1}{2}\gamma; \tag{5}$$

P inside, and P' outside, close to K,

$$\frac{F}{G\sigma}=\Omega'=-4\pi+2\pi\sin\tfrac{1}{2}\gamma. \tag{6}$$

A particle at K, inside or outside, will stick to the surface of the bowl, in stable equilibrium; for if slightly displaced on a small smooth spot at K, it will beat time with a pendulum of length (A. J. M., § 7, p. 386),

$$\frac{g}{G\sigma\pi}\,\frac{KA^3}{OA^2}, \tag{7}$$

this length is $\frac{4}{3}\,\frac{\Delta}{\sigma}\,\frac{KA^3}{OA^2}$ of the earth's radius with a mean density Δ; or the beat is equal to $\sqrt{\left(\frac{1}{3}\,\frac{\Delta}{\sigma}\,\frac{KA^3}{OA^2}\right)}$ of the period of the grazing satellite.

3. In the previous demonstration it was assumed that the bowl was the segment of a sphere made by a plane; but as the result is independent of the size of the segment, it holds true when the segment is made small; and then by summation the result in (1) § 1 is seen to be unaltered in form when the bowl is bounded by any other curve.

This is evident by elementary geometry in fig. 3; the element dS of the spherical surface at E has the potential $\frac{dS}{EP}$ at P, and

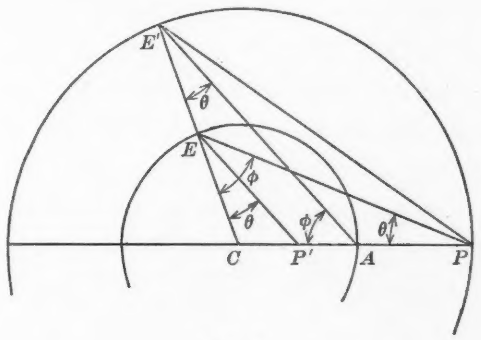

FIG. 3.

$$EP=CE\cos\phi+CP\cos\theta, \tag{1}$$

so that

$$\begin{aligned}\frac{dU}{G\sigma}&=\frac{dS}{EP}=\frac{dS}{EP^2}\,(CE\cos\phi+CP\cos\theta)\\&=CE\,\frac{dS\cos\phi}{EP^2}+CP'\,\frac{dS\cos\theta}{EP'^2}\left(\text{because }\frac{CP}{CP'}=\frac{EP^2}{EP'^2}\right)\\&=CE\cdot d\Omega+CP'\cdot d\Omega',\end{aligned} \tag{2}$$

reckoning Ω, Ω' positive when their aspect from P is the concave side of the bowl; and then by summation,

$$\frac{U}{G\sigma} = c\Omega + r'\Omega', \tag{3}$$

whatever the shape of the boundary rim of the bowl.

And for F, the radial force along PC of attraction at P,

$$\frac{dF}{G\sigma} = \frac{dS\cos\theta}{EP^2} = \frac{CP'}{CP}\frac{dS\cos\theta}{EP'^2} = \frac{r'}{r}\,d\Omega' = \frac{c^2}{r^2}\,d\Omega', \tag{4}$$

$$\frac{F}{G\sigma} = \frac{c^2}{r^2}\,\Omega'. \tag{5}$$

These two or three lines of geometry can thus replace some pages of analysis in Maxwell's E. and M.

For a complete sphere, and

$$P \text{ inside,} \quad \Omega = 4\pi, \ \Omega' = 0, \quad \frac{U}{G\sigma} = 4\pi c, \quad F = 0, \tag{6}$$

$$P \text{ outside,} \ \Omega = 0, \quad \Omega' = 4\pi, \quad \frac{U}{G\sigma} = 4\pi r' = \frac{4\pi c^2}{r}, \quad \frac{F}{G\sigma} = \frac{4\pi c^2}{r^2}, \tag{7}$$

the well-known results for a spherical shell, and thence for a solid sphere given first by Newton in the "Principia," and of pioneering interest in justifying his theory of gravitation.

Because evidence has been found recently, by Prof. J. C. Adams, that Newton laid aside his calculations for nineteen years, till 1684, not only on account of his erroneous estimate of the size of the earth, at sixty land miles to the degree of latitude instead of sixty-nine; but also because Newton wanted to prove that the attraction of a spherical body like the earth on an external body, like an apple at the surface, was the same as if the earth was condensed into a particle at the centre, an assumption good enough for its attraction on a distant body like the moon, but requiring justification for an apple on the surface. As soon as this was clear, he set to work at once on the "Principia."

4. The theorem in § 1 is general, that if any P. F. at P is given as a function of r, $\mu = \cos\phi$, ψ, by $V = V(r, \mu, \psi)$, so that $V' = V\left(r' = \frac{c^2}{r}, \mu, \psi\right)$ is a P. F. at the inverse point P', then

$$r'V(r', \mu, \psi) = \frac{c^2}{r}\,V\left(\frac{c^2}{r}, \mu, \psi\right) \text{ is a P. F. at } P. \tag{1}$$

(W. D. Niven, L. M. S., VIII, 1876, p. 66; W. Burnside, L. M. S., XXV, 1893, p. 99).

The theorem is proved at once by Laplace's operator

$$r^2\frac{d^2}{dr^2}+2r\frac{d}{dr}+\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2}{d\psi^2}, \tag{2}$$

which changes, for $r'=\frac{c^2}{r}$, into

$$r'^2\frac{d^2}{dr'^2}+\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2}{d\psi^2}, \tag{3}$$

and this operator, acting as an annihilator on $r'V'$, reduces to Laplace's equation

$$r'^2\frac{d^2V'}{dr'^2}+2r'\frac{dV'}{dr'}+\frac{d}{d\mu}(1-\mu^2)\frac{dV'}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2V'}{d\psi^2}=0. \tag{4}$$

Or in the algebraical form employed by Niven, if V in $f(V, x, y, z)=0$ satisfies Laplace's operator $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0$, so also does the function

$$f\left(\frac{r}{c}V,\quad \frac{c^2}{r^2}x,\quad \frac{c^2}{r^2}y,\quad \frac{c^2}{r^2}z\right)=0.$$

A combination of V and $\frac{r'}{c}V'$, or $\frac{c}{r}V'$ will then give a P. F., where $V=V'$ on the sphere $r=c=r'$.

5. For a distribution symmetrical about the axis Ox, the P. F. V satisfies Laplace's equation in the form

$$\frac{d}{dx}\left(y\frac{dV}{dx}\right)+\frac{d}{dy}\left(y\frac{dV}{dy}\right)=0; \tag{1}$$

so that a function N can be assigned, the Stokes or stream function (S. F.), such that

$$y\frac{dV}{dx}=\frac{dN}{dy},\quad y\frac{dV}{dy}=-\frac{dN}{dx}, \tag{2}$$

and the meridian curves of the surface of constants V and N are orthogonal.

The factor 2π of N in A. J. M., § 11, introduced by Maxwell has been omitted here, and the sign changed; so that here the slope or gradient of V is changed into the gradient of N by a rotation through a right angle against the clock or sun, *widdershins*. Also

$$\frac{dV}{dx}=\frac{1}{y}\frac{dN}{dy},\quad \frac{dV}{dy}=-\frac{1}{y}\frac{dN}{dx}, \tag{3}$$

$$\frac{d^2V}{dxdy}=\frac{d}{dy}\left(\frac{1}{y}\frac{dN}{dy}\right)=-\frac{d}{dx}\left(\frac{1}{y}\frac{dN}{dx}\right), \tag{4}$$

$$\frac{d}{dx}\left(\frac{1}{y}\frac{dN}{dx}\right)+\frac{d}{dy}\left(\frac{1}{y}\frac{dN}{dy}\right)=0. \tag{5}$$

Better then use the ordinary x, y coordinates instead of z, ϖ, or Maxwell's b, A; because at a large distance from the axis Ox, these equations (1) and (5) degenerate into

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=0, \quad \frac{d^2N}{dx^2}+\frac{d^2N}{dy^2}=0, \tag{6}$$

as for plane orthogonal conjugate functions, V and N.

In polar coordinates in fig. 1, with

$$x = c\cos\gamma - r\cos\phi, \quad y=r\sin\phi, \tag{7}$$

$$\frac{d}{dr}=-\cos\phi\frac{d}{dr}+\sin\phi\frac{d}{dy}, \quad \frac{d}{rd\phi}=\sin\phi\frac{d}{dx}+\cos\phi\frac{d}{dy}, \tag{8}$$

$$\frac{dN}{dr}=-\cos\phi\frac{dN}{dx}+\sin\phi\frac{dN}{dy}=y\cos\phi\frac{dV}{dy}+y\sin\phi\frac{dV}{dx}$$

$$=y\frac{dV}{rd\phi}=\sin\phi\frac{dV}{d\phi}, \tag{9}$$

$$\frac{dN}{rd\phi}=\sin\phi\frac{dN}{dx}+\cos\phi\frac{dN}{dy}=-y\sin\phi\frac{dV}{dy}+y\cos\phi\frac{dV}{dr}$$

$$=-y\frac{dV}{dr}=-\sin\phi\frac{rdV}{dr}, \tag{10}$$

and the V gradient is changed into the N gradient by a rotation through a right angle with the clock and sun, *deasil*.

But here x and $r\cos\phi$ are antagonistic, so that V, *widdershins* with N with respect to x and y, becomes *deasil* in the polar coordinates r, ϕ.

With dn the outward normal element of an equipotential V, and ds the element of the meridian curve of V, normal element of N,

$$\frac{dN}{ds}=y\frac{dV}{dn}=4\pi\sigma y, \quad N_1-N_2=\int 4\pi\sigma y ds, \tag{11}$$

and this is twice the charge or induction through the zone S_1-S_2 on the equipotential V; σ denoting the electrical density of induction.

6. A relation similar to (1) § 4, holds for a Stokes stream function (S. F.) at P; denoting it by N, or $N(r, \mu=\cos\phi)$ and by $N'=N\left(r'=\frac{c^2}{r}, \mu\right)$ at P', then

$$\frac{r}{c}N'=\frac{c}{r'}N' \tag{1}$$

is a S. F. at P; and Laplace's operator in (2) § 4, is replaced for the S. F. by

$$r^2\frac{d^2}{dr^2}+(1-\mu^2)\frac{d^2}{d\mu^2}, \tag{2}$$

changing with $r=\frac{c^2}{r'}$ into

$$r'^2\frac{d^2}{dr'^2}+2r'\frac{d}{dr'}+(1-\mu^2)\frac{d^2}{d\mu^2}, \tag{3}$$

so that, operating as an annihilator on $\frac{cN'}{r'}$,

$$r'^2\frac{d^2N'}{dr'^2}+(1-\mu^2)\frac{d^2N'}{d\mu^2}=0. \tag{4}$$

Thus, if

$$y^2M(r,\mu)=r^2\sin^2\phi M(r,\cos\phi), \tag{5}$$

is a S. F. at P, then $y'^2M(r',\mu)$ is a S. F. at P'; and

$$\frac{r}{c}y'^2M\left(\frac{c^2}{r},\mu\right)=\frac{r}{c}r'^2\sin^2\phi M\left(\frac{c^2}{r},\mu\right)$$
$$=\frac{c^3}{r}\sin^2\phi M\left(\frac{c^2}{r},\mu\right)=\frac{c^3}{r^3}y^2M\left(\frac{c^2}{r},\mu\right) \tag{6}$$

is also a S. F. at P.

Axial differentiation $\frac{d}{dx}$ will give a new P. F. and S. F., such as $\frac{dV}{dx}$ and $\frac{dN}{dx}$; and since, in Maxwell's notation, with $A=y\cos\psi$,

$$\frac{dV}{dy}=\frac{dV}{dA}\cos\psi=\frac{1}{A}\frac{dN}{db}\cos\psi; \tag{7}$$

this transverse differentiation will give a new P. F., tesseral of the first order, suitable for use in a uniform field perpendicular to the axis Ox.

7. For a plane circular plate AB, not dished as a bowl, $c=\infty$, $\Omega'=0$, and the result in (1) § 1, changes, in Maxwell's coordinates A, b in E. and M. § 701, to the expression in A. J. M., § 3, p. 378.

$$\frac{W}{G\sigma}=aP-AQ-b\Omega, \tag{1}$$

$$\frac{1}{G\sigma}\left(\frac{dW}{da},\ \frac{dW}{dA},\ \frac{dW}{db}\right)=P,\ -Q,\ -\Omega; \tag{2}$$

illustrating the application of the complete Elliptic Integral, First, Second and Third, in P, Q, Ω; as P here represents the potential of the rim of the plate, Q and Ω the magnetic potential for uniform magnetization, in the plate and normal to it.

For any other boundary of a plane plate the normal component of the attraction, or the magnetic potential of uniform normal magnetization, is still given at a point P by the conical angle Ω of the plate as seen from P.

For if $d\alpha$ denotes a small element of area round a point Q on the plate, the normal component of the attraction of the element is

$$\frac{G\sigma d\alpha}{PQ^2}\cos(\text{angle between } PQ \text{ and normal}) = G\sigma d\Omega \tag{3}$$

and so in $G\sigma\Omega$ for the whole area.

Thus, for an infinite plate, $\Omega = 2\pi$, and the field of the attraction is uniform and $2\pi G\sigma$; changing to $-2\pi G\sigma$ in crossing the plate, a total change of $4\pi G\sigma$, in accordance with a general theorem.

The S. F. L of the plate, with the sign changed to that in A. J. M., § 12, p. 391, and omitting 2π, is then given by

$$\frac{L}{G\sigma} = \tfrac{1}{2}abP + \tfrac{1}{2}bAQ + \tfrac{1}{2}(a^2 - A^2)\Omega, \tag{4}$$

$$\frac{dL}{dA} = A\frac{dU}{db} = -G\sigma A\Omega, \quad \frac{dL}{db} = -A\frac{dU}{dA} = G\sigma AQ, \tag{5}$$

as in § 3; and the S. F. of P, the rim P. F., is $bP + a\Omega$.

8. Here, as in A. J. M., § 20, p. 405, for the flat circular plate, with $PQ = r$, $\theta = 2\omega$, $r^2 = r_1^2\cos^2\omega + r_2^2\sin^2\omega$,

$$\begin{aligned} Q &= \int_0^{2\pi}\frac{-a\cos\theta d\theta}{r} = \frac{8a}{r_1+r_2}\frac{K-E(\varkappa)}{\varkappa}, \quad P-Q = \int\frac{a(1+\cos\theta)\,d\theta}{r} \\ &= \frac{8a}{r_1}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\omega d\omega}{\Delta\omega} = \frac{8a}{r_1}\frac{E(\gamma)-\gamma'^2G}{\gamma^2} = \frac{8a}{r_1+r_2}\frac{E(\varkappa)-(1-\varkappa)K}{\varkappa}, \end{aligned} \tag{1}$$

$$\begin{aligned} P &= \int_0^{2\pi}\frac{ad\theta}{r} = \int_0^{\frac{1}{2}\pi}\frac{4ad\omega}{\sqrt{(r_1^2\cos^2\omega + r_2^2\sin^2\omega)}} = \int_{r_2}^{r_1}\frac{4adr}{\sqrt{(r_1^2-r^2\cdot r^2-r_2^2)}} \\ &= \frac{4aG}{r_1} = \frac{8aK}{r_1+r_2} \rightarrow \frac{2\pi a}{R}, \end{aligned} \tag{2}$$

where G, K is the complete quarter period to comodulus $\gamma' = \frac{r_2}{r_1}$, or modulus $\varkappa = \frac{r_1-r_2}{r_1+r_2}$, and R is Gauss's arithmetic-geometric mean (A. G. M) of r_1 and r_2.

Inserting some further intermediate values of the series of quadric transformations, such as

$$\lambda=\frac{1-\varkappa'}{1+\varkappa'}=\left(\frac{\sqrt{r_1}-\sqrt{r_2}}{\sqrt{r_1}+\sqrt{r_2}}\right)^2,\quad K=(1+\lambda)L=\frac{r_1+r_2}{\frac{1}{2}(\sqrt{r_1}+\sqrt{r_2})^2}L;$$

$$\mu=\frac{1-\lambda'}{1+\lambda'}=\left(\frac{\sqrt{\frac{r_1+r_2}{2}}-\sqrt[4]{r_1r_2}}{\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}}\right)^2,\quad L=(1+\mu)M=\frac{(\sqrt{r_1}+\sqrt{r_2})^2}{\frac{1}{2}\left(\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}\right)^2}M;$$

$$\nu=\frac{1-\mu'}{1+\mu'}=\left(\frac{\frac{\sqrt{r_1}+\sqrt{r_2}}{2}-\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}}{\frac{\sqrt{r_1}+\sqrt{r_2}}{2}+\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}}\right)^2,$$

$$M=(1+\nu)N=\frac{\left(\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}\right)^2}{\frac{1}{2}\left(\frac{\sqrt{r_1}+\sqrt{r_2}}{2}+\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}\right)^2}N; \tag{3}$$

the ring potential of a mass m round the circle AB is given by

$$\frac{m}{r_1}\frac{G}{\frac{1}{2}\pi}=\frac{m}{\frac{1}{2}(r_1+r_2)}\frac{K}{\frac{1}{2}\pi}=\frac{m}{\frac{1}{4}(\sqrt{r_1}+\sqrt{r_2})^2}\frac{L}{\frac{1}{2}\pi}=\frac{m}{\frac{1}{4}\left(\sqrt{\frac{r_1+r_2}{2}}+\sqrt[4]{r_1r_2}\right)^2}\frac{M}{\frac{1}{2}\pi}$$

$$=\frac{m}{\frac{1}{4}\left(\frac{\sqrt{r_1}+\sqrt{r_2}}{2}+\sqrt[8]{r_1r_2}\sqrt[4]{\frac{r_1+r_2}{2}}\right)^2}\frac{N}{\frac{1}{2}\pi}\rightarrow\frac{m}{R} \tag{4}$$

For a point close to the rim, r_2 is small, and r_1, $2A$ may be replaced by $2a$; P is then large, and Q too, but $P-Q$ is finite; and as in A. J. M., § 22, writing it

$$P=\int_0^{\frac{1}{2}\pi}\frac{4a\,d\omega}{r}=\int\frac{4a\sin\omega\,d\omega}{r}+\int\frac{4a(1-\sin\omega)\,d\omega}{r}, \tag{5}$$

the first integral

$$\int_\omega^{\frac{1}{2}\pi}\frac{4a\sin\omega\,d\omega}{r}=\int_{r_2}^{r}\frac{4a\sin\omega\,dr}{\sqrt{(r_1^2-r^2\cdot r^2-r_2^2)}}$$

$$=\frac{4a}{\sqrt{(r_1^2-r_2^2)}}\int\frac{dr}{\sqrt{(r^2-r_2^2)}}=\frac{4a}{\sqrt{(r_1^2-r_2^2)}}\operatorname{ch}^{-1}\frac{r}{r_2}, \tag{6}$$

$$\int_0^{\frac{1}{2}\pi}\frac{4a\sin\omega\,d\omega}{r}=2\sqrt{\frac{a}{A}}\operatorname{ch}^{-1}\frac{r_1}{r_2}=2\sqrt{\frac{a}{A}}\log\frac{r_1+\sqrt{(r_1^2-r_2^2)}}{r_2}\rightarrow 2\log\frac{2r_1}{r_2}$$

and the second integral, with $r>r_1\cos\omega$, $\Delta\omega>\cos\omega$,

$$\int_0^{\frac{1}{2}\pi}\frac{4a(1-\sin\omega)\,d\omega}{r}<\frac{4a}{r_1}\int\frac{1-\sin\omega}{\cos\omega}d\omega \text{ or } 2\int\frac{\cos\omega\,d\omega}{1+\sin\omega}=2\log 2, \tag{8}$$

so that we can take

$$P=2\log\frac{4r_1}{r_2}+\text{small terms, making}$$

$$G\to\log\frac{4r_1}{r_2}=\log\frac{4}{\gamma'},\text{ practically, when }\gamma'\text{ is small; and then}$$

$$G=2\log 2\frac{1+\varkappa}{\varkappa'}=2\log\frac{4}{\varkappa'}=2K;\ \text{ or }\gamma'=4e^{-G}=4e^{-\frac{1}{2}\pi\frac{G}{G'}},\ \varkappa'=4e^{-\frac{1}{2}\pi\frac{K}{K'}},$$

as they may be written, with $G'=K'=\frac{1}{2}\pi$; (9)

$$P-Q=\frac{8a}{r_1}\int\frac{\cos^2\omega d\omega}{\Delta\omega}<\frac{8a}{r_1}\int\cos\omega d\omega,\text{ or }4; \tag{10}$$

$$P+Q=2P-4+\text{small terms}=4\log\frac{4r_1}{er_2}+\text{small terms}; \tag{11}$$

and this is large as r_2 is small.

Thus, for example, the capacity of a ring AB, of small circular cross section πc^2 may be taken, with $r_2=c$, $r_1=2a$,

$$\frac{2\pi a}{P}=\frac{\pi a}{\log\frac{8a}{c}}=\frac{\text{circumference of }AB}{\log\dfrac{64\text{ area of circle }AB}{\text{area of cross section}}} \tag{12}$$

For instance, with $a=10c$, the capacity is $\dfrac{\pi a}{\log_e 80}=\dfrac{\pi a}{4.382}=0.7168a$.

So also for the potential of a circular plate, in exact functions tabulated numerically, instead of in an approximation by series, as in Thomson and Tait, § 546.

9. When the bowl in fig. 1 is insulated and electrified, the electrical potential can be written, in analogy with the potential of the bowl itself in (1), § 1,

(W. Thomson, Liouville, Oct. 8, 1845, "Electrical Papers," XVIII, p. 178; J. C. Maxwell, "Scientific Papers," II, p. 303; Ferrers, Q. J. M., XVIII, 1881, p. 97; Gallop, Q. J. M., XXI, 1886, p. 229.)

$$V=\omega+\frac{c}{r}\omega'=\omega+\frac{r'}{c}\omega',\quad cV=c\omega+r'\omega', \tag{1}$$

where, in fig. 1, ω and ω' are plane angles, given by

$$\sin\omega=\frac{2a}{r_1+r_2},\quad \sin\omega'=\frac{2a}{r_1'+r_2'}=\frac{2r\sin\gamma}{r_1+r_2}, \tag{2}$$

$$r_1=PA,\quad r_2=PB,\quad r_1'=P'A,\quad r_2'=P'B,\quad OP=r,\quad OP'=r'=\frac{c^2}{r}, \tag{3}$$

$$AB=2a,\quad ACB=2\gamma,\quad a=c\sin\gamma,\quad OC=c\cos\gamma,\quad OA=c, \tag{4}$$

$$\frac{r_1'}{r}=\frac{AP'}{AP}=\frac{DP'}{DP}=\frac{r'-c}{c-r}=\frac{r'}{c}=\frac{c}{r}=\frac{r_2'}{r_2}=\frac{r_1'+r_2'}{r_1+r_2}=\frac{\sin\omega}{\sin\omega'}, \tag{5}$$

the ratio of the line elements or relative magnification at P and P'; and $\sin\omega$, $\sin\omega'$ is the excentricity of the ellipse with foci at A, B, passing through P and P'.

The statement in (1) is verified, because ω and $\frac{c}{r}\omega'$ are P. F.'s at P; and over the bowl AKB, $r=r'=c$, $r_1'=r_1$, $r_2'=r_2$, $\sin\omega'=\sin\omega$, $\omega'=\pi-\omega$, so that $V=\pi$; while $\omega=\omega'$, $V=2\omega$ over the remaining part $AK'B$ of the spherical surface. At infinity, $r_1=r_2=r=\infty$, $r'=0$,

$$\omega=\sin\omega=\frac{a}{r}=\frac{c\sin\gamma}{r},\ \sin\omega'=\sin\gamma,\ V(\infty)=\frac{c\sin\gamma}{r}+\frac{c\gamma}{r}=\frac{E}{r}, \tag{6}$$

so that the charge $E=c(\gamma+\sin\gamma)$.

The term ω in V is the potential of the electrification of the flat circular disc AB, insulated and at potential $\frac{1}{2}\pi$; but the term $\frac{c}{r}\omega'$ is obtained by inversion of the disc AB with respect to C, as the potential of a bowl AHB on the base AB, part of a spherical surface passing through C, when this bowl is earthed and influenced by a point charge $-\frac{1}{2}\pi c$ at C; because $\omega'=\frac{1}{2}\pi$ over AHB.

The sum of the two terms is then the electric potential of the insulated bowl on the base AB, centre at C.

10. The difference of the two terms

$$V'=-\omega+\frac{c}{r}\omega' \tag{1}$$

is also a P. F., zero over the spherical bowl $AK'B$, where $r=c$, $\omega'-\omega=0$; but over AKB,

$$\omega'+\omega=\pi,\quad V=\pi-2\omega=2\omega'-\pi. \tag{2}$$

At infinity, $r_1=r_2=r=\infty$, $r'=0$;

$$\omega=\sin\omega=\frac{a}{\gamma}=\frac{c\sin\gamma}{r},\quad \sin\omega'=\sin\gamma,\quad V'=\frac{c}{r}(\gamma-\sin\gamma), \tag{3}$$

so that the charge is $c(\gamma-\sin\gamma)$.

At the centre C, where $r=0$, $\omega=\gamma$,

$$\frac{c}{r}\omega'=\sin\gamma,\quad V=-\gamma+\sin\gamma. \tag{4}$$

Mr. J. R. Wilton gives (*Messenger of Mathematics*, p. 96, August, 1914),

$$\phi=-\omega+\frac{c}{r}(\pi-\omega') \tag{5}$$

as the P. F. of the bowl AKB, uninsulated, in presence of a point charge πc at the centre C.

11. The S. F. A of the P. F. ω is then found to be given by

$$A=\tfrac{1}{2}\sqrt{[AB^2-(PA-PB)^2]}=\sqrt{(PA\cdot PB)}\sin\tfrac{1}{2}APB, \tag{1}$$

so that A is the semi-conjugate axis of the confocal hyperbola through P, and the meridian curves of constant ω and A are confocal ellipses and hyperbolas.

More generally, any oblate spheroid of which the disc AB is the focal circle, if insulated and electrified with a charge E, will have a P. F. V and S. F. N given at an external point P in the meridian plane APB by

$$V=\frac{2E}{AB}\sin^{-1}\frac{AB}{PA+PB},\quad N=E\sqrt{\left[1-\left(\frac{PA-PB}{AB}\right)^2\right]} \tag{2}$$

as this verifies at infinity, where $PA=PB=PO=$ infinity, and

$$V=\frac{2E}{AB}\sin^{-1}\frac{AB}{2PO}=\frac{2E}{AB}\cdot\frac{AB}{2PO}=\frac{E}{PO}. \tag{3}$$

The electrical density σ at a point Q on the spheroid will be given by

$$\sigma=\frac{E}{2\pi}\cdot\frac{1}{QA+QB}\cdot\frac{1}{\sqrt{(QA\cdot QB)}}=\frac{\text{electrical force}}{4\pi}, \tag{4}$$

$$\text{electrical force}=4\pi\sigma=\frac{E}{\tfrac{1}{2}(QA+QB)\sqrt{(QA\cdot QB)}}, \tag{5}$$

electrical charge on the zone $QK=$ half the difference of

the S. F. at K and $Q=\tfrac{1}{2}E\left\{1-\sqrt{\left[1-\left(\frac{QA-QB}{AB}\right)^2\right]}\right\}$. $\quad$ (6)

12. Putting

$$\sqrt{\left[\left(\frac{r_1+r_2}{2}\right)^2-a^2\right]}=B,\ \text{with}\sqrt{\left[a^2-\left(\frac{r_1-r_2}{2}\right)^2\right]}=A, \tag{1}$$

as before in (1) § 11, where

$$r_1^2=x^2+(y+a)^2,\quad r_2^2=x^2+(y-a)^2,\quad r_1^2-r_2^2=4ay, \tag{2}$$

$$\frac{dr_1}{dx}=\frac{x}{r_1},\quad \frac{dr_2}{dx}=\frac{x}{r_2},\quad \frac{dr_1}{dy}=\frac{y+a}{r_1},\quad \frac{dr_2}{dy}=\frac{y-a}{r_2}, \tag{3}$$

$$B=\sqrt{(\tfrac{1}{2}\cdot r_1r_2+x^2+y^2-a^2)},\quad A=\sqrt{(\tfrac{1}{2}\cdot r_1r_2-x^2-y^2+a^2)}, \tag{4}$$

$$B^2+A^2=r_1r_2,\quad B^2-A^2=x^2+y^2-a^2,\quad AB=ax, \tag{5}$$

$$A,\ B=\tfrac{1}{2}\sqrt{(r_1r_2+2ax)}\mp\tfrac{1}{2}\sqrt{(r_1r_2-2ax)}, \tag{6}$$

$$\omega=\sin^{-1}\frac{2a}{r_1+r_2}=\cos^{-1}\frac{2B}{r_1+r_2}, \tag{7}$$

$$\frac{d\omega}{dx}=\frac{-\dfrac{2a}{r_1+r_2}\left(\dfrac{dr_1}{dx}+\dfrac{dr_2}{dx}\right)}{2B}=-\frac{ax}{Br_1r_2}=-\frac{A}{r_1r_2} \tag{8}$$

$$\frac{dA}{dy}=\frac{-\frac{1}{4}(r_1-r_2)\left(\frac{dr_1}{dy}-\frac{dr_2}{dy}\right)}{A}=\frac{-(r_1-r_2)\left(\frac{y+a}{r_1}-\frac{y-a}{r_2}\right)}{4A}$$

$$=\frac{y(r_1-r_2)^2-a(r_1^2-r_2^2)}{4Ar_1r_2}=\frac{y(a^2-A^2)-a^2y}{Ar_1r_2}=-\frac{yA}{r_1r_2}=y\frac{d\omega}{dx}, \quad (9)$$

$$\frac{d\omega}{dy}=\frac{\frac{-2a}{r_1+r_2}\left(\frac{y+a}{r_1}+\frac{y-a}{r_2}\right)}{2B}=\frac{-ay(r_1+r_2)+a^2(r_1-r_2)}{Br_1r_2(r_1+r_2)}$$

$$=\frac{-ay(r_1^2-r_2^2)+a^2(r_1-r_2)^2}{Br_1r_2(r_1^2-r_2^2)}=\frac{-a^2y^2+a^2(a^2-A^2)}{Br_1r_2ay}$$

$$=\frac{ax^2-aB^2}{Br_1r_2y}=\frac{xAB-aB^2}{Br_1r_2y}=\frac{xA-aB}{r_1r_2y}, \quad (10)$$

$$\frac{dA}{dx}=\frac{-\frac{1}{4}(r_1-r_2)\left(\frac{x}{r_1}-\frac{x}{r_2}\right)}{A}=\frac{x(r_1-r_2)^2}{4Ar_1r_2}$$

$$=\frac{x(-A^2+a^2)}{Ar_1r_2}=\frac{-xA^2+aAB}{Ar_1r_2}=\frac{-xA+aB}{r_1r_2}=-y\frac{d\omega}{dy} \quad (11)$$

which proves, as defined in (2), § 5, that A is the S. F. of the P. F. ω, as stated above in § 11.

Similarly, we prove that B is the S. F. of the P. F.

$$\omega_1=\mathrm{ch}^{-1}\frac{2a}{r_1-r_2}=\mathrm{sh}^{-1}\frac{2A}{r_1-r_2}, \quad (12)$$

and ω_1 can be the P. F. of the electrification of the infinite plate with the circular hole AB cut out, or of any confocal hyperboloid of revolution, the electric charge being infinite.

Then B is the semi-minor axis of the confocal ellipse through P, and

$$B=a\cot\omega=x\coth\omega_1=\sqrt{(r_1r_2)}\cos\tfrac{1}{2}APB, \quad (13)$$

$$A=x\tan\omega=a\ \ \mathrm{th}\,\omega_1=\sqrt{(r_1r_2)}\sin\tfrac{1}{2}APB; \quad (14)$$

these relations help to settle a doubtful sign.

13. The S. F. at P of the P. F. ω being

$$A=\sqrt{(r_1r_2)}\sin\tfrac{1}{2}APB=\sqrt{\left[a^2-\left(\frac{r_1-r_2}{2}\right)^2\right]}, \quad (1)$$

as proved by the preceding differentiations, the S. F. at P' of the P. F. ω' is

$$A'=\sqrt{(r_1'r_2')}\sin\tfrac{1}{2}AP'B=\sqrt{\left[a^2-\left(\frac{r_1'-r_2'}{2}\right)^2\right]}$$

$$=\sqrt{\left[c^2\sin^2\gamma-\frac{c^2}{r^2}\left(\frac{r_1-r_2}{2}\right)^2\right]}=\frac{c}{r}\sqrt{\left[r^2\sin^2\gamma-\left(\frac{r_1-r_2}{2}\right)^2\right]}, \quad (2)$$

and then by (1), § 6,

$$C=\sqrt{(r_1r_2)}\sin\tfrac{1}{2}AP'B=\sqrt{\left[r^2\sin^2\gamma-\left(\frac{r_1-r_2}{2}\right)^2\right]} \tag{3}$$

is a S. F. at P.

But we must not suppose that it has the P. F. $\frac{c}{r}\omega'$; the S. F. of this P. F. $\frac{c}{r}\omega'$ must be determined to have the S.F. by addition of the whole P.F. $\omega+\frac{c}{r}\omega'$.

By analogy with the S. F. of the material of the bowl, given in A. J. M., § 12, where, with the sign changed, AQ is the S. F. of the P. F. Ω, satisfying the relations of A. J. M., § 13,

$$\frac{dAQ}{dA}=-A\frac{d\Omega}{db},\quad \frac{dAQ}{db}=A\frac{d\Omega}{dA}, \tag{4}$$

and so there the P. F. $\frac{c}{r}\Omega'$ has the S.F. $aP-c\Omega\cos V-c\Omega'\cos\phi$; the analogy shows that the S. F. of the P. F. $\frac{c}{r}\omega'$ will contain a term $c\omega'\cos\phi$, and Mr. Wilton has found (*Messenger of Mathematics*, p. 70, August, 1914), that the complete expression is

$$c\omega-c\omega'\cos\phi, \text{ the S. F. of the P. F. } \frac{c}{r}\omega'. \tag{5}$$

Thus the P. F.

$$V=\omega+\frac{c}{r}\omega' \tag{6}$$

has the S. F.

$$N=A+c\omega-c\omega'\cos\phi. \tag{7}$$

This is verified by the differentiation

$$\begin{aligned}\frac{dN}{dx}+y\frac{dV}{dy}&=\frac{dA}{dx}+y\frac{d\omega}{dy}+c\frac{d\omega}{dx}-c\frac{d\omega'}{dx}\cos\phi+c\omega'\sin\phi\frac{\sin\phi}{r}\\&\qquad+r\sin\phi\left(\frac{c}{r}\frac{d\omega'}{dy}-\frac{c}{r^2}\omega'\sin\phi\right)\\&=c\frac{d\omega}{dx}+c\left(-\frac{d\omega'}{dx}\cos\phi+\frac{d\omega'}{dy}\sin\phi\right)=c\left(\frac{d\omega}{dx}+\frac{d\omega'}{dr}\right)=0,\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{dN}{dy}-y\frac{dV}{dx}&=\frac{dA}{dy}-y\frac{d\omega}{dx}+c\frac{d\omega}{dy}-c\frac{d\omega'}{dy}\cos\phi+c\omega'\sin\phi\frac{\cos\phi}{r}\\&\qquad-r\sin\phi\left(\frac{c}{r}\frac{d\omega'}{dx}+\frac{c}{r^2}\omega'\cos\phi\right)\\&=c\frac{d\omega}{dy}-c\left(\frac{d\omega'}{dx}\sin\phi+\frac{d\omega'}{dy}\cos\phi\right)=c\left(\frac{d\omega}{dy}-\frac{d\omega'}{rd\phi}\right)=0.\end{aligned} \tag{9}$$

These relations in (8) (9) are verified by putting

$$B'=\sqrt{\left[\left(\frac{r_1'+r_2'}{2}\right)^2-a^2\right]}=\frac{c}{r}D,\ \text{with}\ A'=\sqrt{\left[a^2-\left(\frac{r_1'-r_2'}{2}\right)^2\right]}=\frac{c}{r}C, \tag{10}$$

as before; and then, differentiating,

$$\frac{d\omega'}{rd\phi}=\frac{r'}{r}\frac{d\omega'}{r'd\phi}=\frac{c^2}{r^2}\left(\frac{d\omega'}{dx'}\sin\phi+\frac{d\omega'}{dy'}\cos\phi\right)$$

$$=\frac{c^2}{r^2}\left(\frac{-A'\sin\phi}{r_1'r_2'}+\frac{x'A'-aB'}{r_1'r_2'y'}\cos\phi\right)=-\frac{A'\sin\phi}{r_1r_2}+\frac{x'A'-aB'}{r_1r_2\frac{c^2}{r^2}y}\cos\phi$$

$$=-\frac{\frac{c}{r}C\sin\phi}{r_1r_2}+\frac{\left(c\cos\gamma-\frac{c^2}{r}\cos\phi\right)\frac{c}{r}C\cos\phi-c\sin\gamma\frac{c}{r}D\cos\phi}{r_1r_2\frac{c^2}{r^2}y}$$

$$=\left[-\frac{c}{r}Cr\sin^2\phi+\left(c\cos\gamma-\frac{c^2}{r}\cos\phi\right)\frac{r}{c}C\cos\phi-c\sin\gamma\frac{r}{c}D\cos\phi\right]\div r_1r_2y$$

$$=[(-c+r\cos\phi\cos\gamma)C-Dr\sin\gamma\cos\phi]\div r_1r_2y$$

$$=[x(-C\cos\gamma+D\sin\gamma)-a(C\sin\gamma+D\cos\gamma)]\div r_1r_2y$$

$$=\frac{xA-aB}{r_1r_2y}=\frac{d\omega}{dy}; \tag{11}$$

because AE, BE bisect the angles PAP', PBP' if the spherical surface cuts CPP' in E; and $(APC, AEC, AP'C)$ $(BPC, BEC, BP'C)$ are angles in arithmetical progression, and so is their difference, so that

$$\tfrac{1}{2}APB+\tfrac{1}{2}AP'B=AEB=\gamma, \tag{12}$$

$$A=\sqrt{(r_1r_2)}\sin\tfrac{1}{2}APB=\sqrt{(r_1r_2)}\sin(\gamma-\tfrac{1}{2}AP'B)=D\sin\gamma-C\cos\gamma, \tag{13}$$

$$B=\sqrt{(r_1r_2)}\cos\tfrac{1}{2}APB=\sqrt{(r_1r_2)}\cos(\gamma-\tfrac{1}{2}APB)=D\cos\gamma+C\sin\gamma, \tag{14}$$

as if A, B and C, D were orthogonal components of the same vector.

Differentiating again

$$\frac{d\omega'}{dr}=-\frac{r'}{r}\frac{d\omega'}{dr'}=-\frac{c^2}{r^2}\left(-\frac{d\omega'}{dx'}\cos\phi+\frac{d\omega'}{dy'}\sin\phi\right)$$

$$=-\frac{c^2}{r^2}\left(\frac{A'\cos\phi}{r_1'r_2'}+\frac{x'A'-aB'}{r_1'r_2'y'}\sin\phi\right)$$

$$=-\frac{\frac{c}{r}C\cos\phi}{r_1r_2}-\frac{\left(c\cos\gamma-\frac{c^2}{r}\cos\phi\right)\frac{c}{r}C\sin\phi-c\sin\gamma\frac{c}{r}D\sin\phi}{r_1r_2\frac{c^2}{r}\sin\phi}$$

$$=\left[-\frac{c}{r}C\cos\phi-\left(\cos\gamma-\frac{c}{r}\cos\phi\right)C+D\sin\gamma\right]\div r_1r_2$$

$$=\frac{-C\cos\gamma+D\sin\gamma}{r_1r_2}=\frac{A}{r_1r_2}=-\frac{d\omega}{dx}. \tag{15}$$

Mr. Wilton has considered also the S. F.

$$a\frac{\cos\omega+\cos\omega'}{\sin\omega}=B+D, \tag{16}$$

which vanishes over the bowl, and has the P. F.

$$\omega\sin\gamma+\omega_1(1+\cos\gamma), \tag{17}$$

and investigated the physical interpretation. So also

$$a\frac{\operatorname{sh}\omega_1+\operatorname{sh}\omega_1'}{\operatorname{ch}\omega_1}=A+C, \tag{18}$$

is a S. F. with a P. F.

$$\omega(1-\cos\gamma)+\omega_1\sin\gamma. \tag{19}$$

Other combinations can be made and interpreted (J. R. Wilton, *Messenger of Mathematics*, p. 75, August, 1914).

14. Over the bowl AKP_iB,

$$r=c,\quad r_1=2c\sin\tfrac{1}{2}(\gamma+\phi),\quad r_2=2c\sin\tfrac{1}{2}(\gamma-\phi),\quad r_1+r_2=4c\sin\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\phi, \tag{1}$$

$$\sin\omega=\frac{2a}{r_1+r_2}=\frac{2c\sin\gamma}{4c\sin\frac{1}{2}\gamma\cos\frac{1}{2}\phi}=\frac{\cos\frac{1}{2}\gamma}{\cos\frac{1}{2}\phi}=\frac{BK'}{PK'}. \tag{2}$$

But over $BK'A$, the rest of the spherical surface,

$$r_1=2c\sin\tfrac{1}{2}(\phi+\gamma),\quad r_2=2c\sin\tfrac{1}{2}(\phi-\gamma),\quad r_1+r_2=4c\cos\tfrac{1}{2}\gamma\sin\tfrac{1}{2}\phi \tag{3}$$

$$\sin\omega=\frac{\sin\frac{1}{2}\gamma}{\sin\frac{1}{2}\phi}=\frac{AK}{QK}. \tag{4}$$

The P. F. $V=\omega+\dfrac{c}{r}\omega'$ must be adjusted as a single valued function, although ω and ω' are multiple-valued.

Starting from a point S on the spherical surface $AK'B$, where

$$\omega=\omega'=\sin^{-1}\frac{AK}{SK}, \tag{5}$$

taken as the acute angle, and $V=2\omega$, let the point P travel from S to P_i on the interior of the bowl, and for simplicity along the orthogonal circle, with limiting points A and B.

Then P' will travel from S along this circle in the opposite direction, and will reach a point P_0 on the outside of the bowl, coincident with P_i. In this path P' does not cross the base AB of the bowl, and ω' never reaches $\frac{1}{2}\pi$, and $\cos\omega'$ remains positive.

But P in crossing the base AB makes $\omega=\frac{1}{2}\pi$; and beyond AB and up to P_i we must take $\omega>\frac{1}{2}\pi$, and $\cos\omega$ negative.

Thus if ω_0, ω_i denotes the value of ω according as it is reached from S by a path to the outside at P_0, or inside at P_i of the bowl,

$$\sin\omega_0=\sin\omega_i=\frac{BK'}{PK'},\quad \cos\omega_0=-\cos\omega_i,\quad \omega_0+\omega_i=\pi, \tag{6}$$

making π the potential on the bowl; and we can put

$$\omega_0=\tfrac{1}{2}(\pi-\theta),\quad \omega_i=\tfrac{1}{2}(\pi+\theta),\quad \cos\tfrac{1}{2}\theta=\frac{BK'}{PK'}, \tag{7}$$

where $\frac{1}{2}\theta$ is a positive acute angle. At S, the S. F. is

$$N(S)=\tfrac{1}{2}\sqrt{[AB^2-(SA-SB)^2]}+c(1-\cos\phi)\sin^{-1}\frac{AK}{SK}; \tag{8}$$

and travelling from S to P_i,

$$N(P_i)=A+c\omega_i-c\omega_0\cos\phi=\tfrac{1}{2}\sqrt{[AB^2-(PA-PB)^2]}$$
$$+\tfrac{1}{2}(\pi+\theta)c-\tfrac{1}{2}(\pi-\theta)c\cos\phi. \tag{9}$$

At B, $\theta=0$, $A=0$, $\phi=\gamma$, $N(B)=\frac{1}{2}\pi c(1-\cos\gamma)$, (10)

$$N(P_i)-N(B)=A+\tfrac{1}{2}(\pi+\theta)c-\tfrac{1}{2}(\pi-\theta)c\cos\phi-\tfrac{1}{2}\pi c(1-\cos\gamma). \tag{11}$$

At K, $\theta=\gamma$, $\phi=0$, $A=a=c\sin\gamma$,

$$N(K)=c\sin\gamma+\tfrac{1}{2}(\pi+\gamma)c-\tfrac{1}{2}(\pi-\gamma)c=c\sin\gamma+c\gamma, \tag{12}$$

$$N(K_i)-N(B)=c\sin\gamma+c\gamma+\tfrac{1}{2}\pi c(1-\cos\gamma). \tag{13}$$

15. Denoting by σ_i the electrical density at P_i, on the inside of the bowl,

$$4\pi\sigma_i=+\frac{dV_i}{dr}=-\frac{1}{y}\frac{dN(P_i)}{cd\phi} \tag{1}$$

and the charge on the interior of the bowl swept out by the revolution of the arc P_iB is

$$\int 2\pi\sigma_i y c d\phi=\tfrac{1}{2}N(P_i)-\tfrac{1}{2}N(B), \tag{2}$$

and the total charge on the interior of the bowl is

$$\tfrac{1}{2}N(K_i)-\tfrac{1}{2}N(B)=\tfrac{1}{2}(c\sin\gamma+c\gamma)-\tfrac{1}{4}\pi c(1-\cos\gamma). \tag{3}$$

At P_0 on the exterior of the bowl the sign of A must be changed in the S. F., and ω_0, ω_i interchanged, making

$$N(P_0)=-A+c\omega_0-c\omega_i\cos\phi$$
$$=-\tfrac{1}{2}\sqrt{[AB^2-(PA-PB)^2]}+\tfrac{1}{2}(\pi-\theta)c-\tfrac{1}{2}(\pi+\theta)c\cos\phi, \tag{4}$$

$$N(K_0)=-c\sin\gamma-c\gamma. \tag{5}$$

The electrical density σ_0 on the outside of the bowl, at P_0, is then given by

$$4\pi\sigma_0=-\frac{dV_0}{dr}=\frac{1}{y}\frac{dN(P_0)}{cd\phi}, \tag{6}$$

and the charge on the outside of the part of the bowl swept out by P_0B is

$$\int 2\pi\sigma_0 y c d\phi=\tfrac{1}{2}N(B)-\tfrac{1}{2}N(P_0), \tag{7}$$

and the total charge on the outside is

$$\tfrac{1}{2}N(B)-\tfrac{1}{2}N(K_0)=\tfrac{1}{2}(c\gamma+c\sin\gamma)+\tfrac{1}{4}\pi c(1-\cos\gamma). \tag{8}$$

The total charge on the bowl, outside and inside, is then

$$\tfrac{1}{2}N(K_0)-\tfrac{1}{2}N(K_i)=c\gamma+c\sin\gamma \tag{9}$$

at potential π, so that the capacity of the bowl is

$$\frac{c\gamma+c\sin\gamma}{\pi}=\frac{\text{arc } AKB+\text{chord } AOB}{2\pi}=\frac{\text{meridian girth}}{2\pi}, \tag{10}$$

this is the radius of a sphere of the same girth.

This verifies for the complete sphere, and a flat circular disc AB.

The difference of the charge on the part of the bowl swept out by KP, outside and inside, is

$$\begin{aligned}\int 2\pi(\sigma_0-\sigma_i)ycd\phi &= \tfrac{1}{2}N(P_0)-\tfrac{1}{2}N(K_0)+\tfrac{1}{2}N(K_i)-\tfrac{1}{2}N(P_i)\\ &=\tfrac{1}{2}(\pi-\theta)c-\tfrac{1}{2}(\pi+\theta)c\cos\phi-\tfrac{1}{2}(\pi-\theta)c\cos\phi+\tfrac{1}{2}(\pi+\theta)c\\ &=\pi c(1-\cos\phi)=\frac{\text{surface of the part}}{2c},\end{aligned} \tag{11}$$

so that the difference $\sigma_0-\sigma_i$ is constant $=\dfrac{1}{4c}$, $\quad 4\pi(\sigma_0-\sigma_i)=\dfrac{\pi}{c}$. (12)

16. At P_i on the bowl

$$\begin{aligned}A&=\tfrac{1}{2}\sqrt{(4c^2\sin^2\gamma-16c^2\cos^2\tfrac{1}{2}\gamma\sin^2\tfrac{1}{2}\phi)}\\ &=2\cos^2\tfrac{1}{2}\gamma\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)},\end{aligned} \tag{1}$$

$$\omega_i=\tfrac{1}{2}(\pi+\theta),\quad \omega_0=\tfrac{1}{2}(\pi-\theta), \tag{2}$$

$$\frac{d\omega_i}{d\phi}=-\frac{d\omega_0}{d\phi}=\tfrac{1}{2}\frac{d\theta}{d\phi}=\frac{\tfrac{1}{2}\cos\tfrac{1}{2}\gamma\tan\tfrac{1}{2}\phi}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}, \tag{3}$$

$$\cos\tfrac{1}{2}\theta=\frac{\cos\tfrac{1}{2}\gamma}{\cos\tfrac{1}{2}\phi},\quad \sin\tfrac{1}{2}\theta=\frac{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}{\cos\tfrac{1}{2}\phi}, \tag{4}$$

$$\tan\omega_0=\cot\tfrac{1}{2}\theta=\frac{\cos\tfrac{1}{2}\gamma}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}, \tag{5}$$

$$N(P_i)=2c\cos\tfrac{1}{2}\gamma\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}+c\omega_i-c\omega_0\cos\phi, \tag{6}$$

$$\begin{aligned}\frac{dN(P_i)}{cd\phi}&=-\frac{\tfrac{1}{2}\cos\tfrac{1}{2}\gamma\sin\phi}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}-\tfrac{1}{2}(1+\cos\phi)\frac{d\theta}{d\phi}+\omega_0\sin\phi\\ &=-\frac{\cos\tfrac{1}{2}\gamma\sin\phi}{\sqrt{(\cos^2\tfrac{1}{2}\phi-\cos^2\tfrac{1}{2}\gamma)}}+\omega_0\sin\phi=-(\tan\omega_0-\omega_0)\sin\phi,\end{aligned} \tag{7}$$

$$4\pi\sigma_i=-\frac{1}{c\sin\phi}\frac{dN(P_i)}{cd\phi}=\frac{\tan\omega_0-\omega_0}{c}, \tag{8}$$

$$4\pi(\sigma_0-\sigma_i)=\frac{\pi}{c}=\frac{\omega_i+\omega_0}{c}, \tag{9}$$

$$4\pi\sigma_0=\frac{\omega_i+\tan\omega_0}{c}=\frac{\omega_i-\tan\omega_i}{c}. \tag{10}$$

If the potential is U, instead of π, these values of σ and the charge in (9), § 15, must be multiplied by $\frac{U}{\pi}$, making

$$4\pi\sigma_i = \frac{U}{\pi c}(\tan\omega_0 - \omega_0), \quad 4\pi\sigma_0 = \frac{U}{\pi c}(\omega_i - \tan\omega_i), \tag{11}$$

$$4\pi\sigma_i = \frac{U}{\pi c}\left[\frac{AK'}{\sqrt{(AK^2 - KP_0^2)}} - \tan^{-1}\frac{AK'}{\sqrt{(AK^2 - KP_0^2)}}\right], \tag{12}$$

$$\tan\omega_0 = -\tan\omega_i = \frac{AK'}{\sqrt{(AK^2 - KP_0^2)}}. \tag{13}$$

Draw the circle, centre K' and radius $K'P_0$, cutting KA in R, R'; then,

$$AR^2 = K'R^2 - K'A^2 = K'P_0^2 - K'A^2 = KA^2 - KP_0^2;$$

$$ARK' = \omega_0, \quad AK'R = \tfrac{1}{2}\theta, \quad RK'R' = \theta. \tag{14}$$

Anywhere on the axis G on the convex side of the bowl,

$$A = 0, \quad \sin\omega = \frac{OA}{AG} = \sin CGA, \quad \sin\omega' = \frac{CG\sin\gamma}{AG} = \sin CAG,$$

$$\omega = CGA, \quad \omega' = \pi - CAG, \quad \phi = 0, \tag{15}$$

$$N(G) = A + c\omega - c\omega' = a + c(CGA + CAG)a - c\gamma = c(\sin\gamma - \gamma), \tag{16}$$

$$N(G') = A + c\omega' - c\omega = a + c\gamma = a(\sin\gamma + \gamma). \tag{17}$$

And anywhere on the axis at H on the concave side of the bowl, beyond C,

$$A = 0, \quad \omega = CHA, \quad \omega' = CAH, \quad \phi = \pi, \tag{18}$$

$$N(H) = a + c\omega + c\omega' = a + c\gamma = c(\sin\gamma + \gamma). \tag{19}$$

Thus N is $a + c\gamma$ on the concave side along the axis HK, and changes to $a - c\gamma$ along the prolongation KG, in crossing the bowl to the convex side.

17. For the hydrodynamical application to the axial motion U of the circular base AB through an infinite liquid, the velocity function (V. F.) V and stream function (S. F.) N are given by

$$V = Ux\frac{\tan\omega - \omega}{\tfrac{1}{2}\pi}, \quad N = \tfrac{1}{2}Uy^2\frac{\omega - \tfrac{1}{2}\sin 2\omega}{\tfrac{1}{2}\pi} \tag{1}$$

at P; and at P',

$$V' = Ux'\frac{\tan\omega' - \omega'}{\tfrac{1}{2}\pi}, \quad N' = \tfrac{1}{2}Uy'^2\frac{\omega' - \tfrac{1}{2}\sin 2\omega'}{\tfrac{1}{2}\pi}; \tag{2}$$

and the combination

$$V + \frac{c}{r}V', \text{ with } N + \frac{r}{c}N', \tag{3}$$

will serve for Basset's expression of the V. F. and S. F. in the liquid motion due to the axial velocity U of the bowl, in agreement with that given in his "Hydrodynamics," I, pp. 153–156.

Fixed in the current, in an axial stream U past the circular base AB,

$$V=-Ux\left(1-\frac{\tan\omega-\omega}{\frac{1}{2}\pi}\right),\quad N=\tfrac{1}{2}Uy^2\left(1-\frac{\omega-\frac{1}{2}\sin 2\omega}{\frac{1}{2}\pi}\right), \tag{4}$$

with $\omega=\frac{1}{2}\pi$, $N=0$ over the disc AB; while the V. F.

$$\frac{2N\cos\psi}{y}=Uy\left(1-\frac{\omega-\frac{1}{2}\sin 2\omega}{\frac{1}{2}\pi}\right)\cos\psi \tag{5}$$

will give the potential of the electric field, uniform and of potential $Uy\cos\psi$ across the axis Ox, when the field is disturbed by the disc AB, earthed to zero potential; with similar extensions when the disc is replaced by the bowl.

If the bowl is to earth in an axial field Ux, the potential of the field is changed to

$$V=U\left(\frac{c\omega\cos\gamma}{\pi}+x\frac{\tan\omega-\omega}{\pi}\right)+U\frac{c}{r}\left(\frac{c\omega'\cos\gamma}{\pi}+x'\frac{\tan\omega'-\omega'}{\pi}\right)$$
$$-U(c\cos\gamma-x). \tag{6}$$

(Basset, I, p. 154; Gallop, Q. J. M., XXI, p. 256), with the S. F.

$$N=U\left(\frac{Ac\cos\gamma}{\pi}+\tfrac{1}{2}y^2\frac{\omega-\frac{1}{2}\sin 2\omega}{\pi}\right)$$
$$+U\frac{r}{c}\left(\frac{A'c\cos\gamma}{\pi}+\tfrac{1}{2}y'^2\frac{\omega'-\frac{1}{2}\sin 2\omega'}{\pi}\right)-\tfrac{1}{2}Uy^2. \tag{7}$$

18. Similar extensions can be made with applications of the conical angle Ω, Ω', instead of plane angles ω, ω'.

With the P. F. of the material of the bowl

$$V=\Omega+\frac{c}{r}\Omega'=\Omega+\frac{r'}{c}\Omega', \tag{1}$$

$V=\Omega+\Omega'=2\Omega$ over the spherical surface $AK'B$; but over the bowl AKB,

$$\Omega_i=2\pi+\Omega_1,\quad \Omega_0=-2\pi+\Omega_1,\quad \Omega+\Omega'=\Omega_i+\Omega_0=2\Omega_1, \tag{2}$$

so that V is not constant over the bowl AKB, or the rest of the sphere.

To obtain a constant potential over the surface, take

$$V=\Omega-\frac{c}{r}\Omega', \tag{3}$$

which is zero over $AK'B$, and 4π over AKB; and this V can serve for the electric potential when the part $AK'B$ of the sphere is to earth, and the other part AKB is changed to potential 4π, the insulation being made perfect along the line of separation of the two parts AKB, $AK'B$.

The associated S. F. can be written down, as the difference of those given in § 13,

$$N=-AQ-aP+c\Omega\cos\gamma+c\Omega'\cos\phi. \tag{4}$$

But along the insulating circle AB, $A=a$, $P+Q=\infty$; and the charge on each bowl is infinite, with an infinite electromotive force across the line of perfect insulation, so here is an electrical version of the old dynamical paradox of an irresistible force, push, or blow, applied to an immovable body.

The question was proposed for the special case of two hemispheres in the *Mathematical Tripos*, II, 1912, and the answer given in a series, but the series is divergent and the result infinite, as it should be.

19. With the potential of the bowl,

$$V=\Omega+\frac{c}{r}\Omega', \tag{1}$$

$$\frac{dV}{dr}=\frac{d\Omega}{dr}+\frac{c}{r}\frac{d\Omega'}{dr}-\frac{c}{r^2}\Omega'=-\frac{c}{r^2}\Omega', \tag{2}$$

$$\frac{dV}{rd\phi}=\frac{d\Omega}{rd\phi}+\frac{c}{r}\frac{d\Omega'}{rd\phi}=-\frac{Q}{r}, \tag{3}$$

$$\frac{dV}{dx}=-\frac{dV}{dr}\cos\phi+\frac{dV}{rd\phi}\sin\phi=\frac{c}{r^2}\Omega'\cos\phi-\frac{Q}{r}\sin\phi; \tag{4}$$

and as this is a P. F. at P, there is another P. F. at P',

$$\frac{c}{r'^2}\Omega\cos\phi-\frac{Q'}{r'}\sin\phi=\frac{r^2}{c^3}(\Omega\cos\phi-Q\sin\phi), \tag{5}$$

or another P. F. at P,

$$\frac{r}{c^2}(\Omega\cos\phi-Q\sin\phi). \tag{6}$$

The associated S. F. can be obtained from

$$\begin{aligned}\frac{dN}{dx}&=-r\sin\phi\frac{dV}{dy}=-r\sin\phi\left(\frac{dV}{dr}\sin\phi+\frac{dV}{rd\phi}\cos\phi\right)\\&=r\sin\phi\left(\frac{c}{r^2}\Omega'\sin\phi+\frac{Q}{r}\cos\phi\right)\\&=\frac{c}{r}\Omega'\sin^2\phi+Q\sin\phi\cos\phi=M_1,\end{aligned} \tag{7}$$

suppose, and there is another S. F. at P,

$$M_2=\frac{r}{c}\left(\frac{c}{r'}\Omega\sin^2\phi+Q'\sin\phi\cos\phi\right)=\frac{r^2}{c^2}(\Omega\sin^2\phi+Q\sin\phi\cos\phi). \tag{8}$$

Then
$$M_2-M_1=\left(\frac{r^2}{c^2}\Omega-\frac{c}{r}\Omega'\right)\sin^2\phi+\left(\frac{r^2}{c^2}-1\right)Q\sin\phi\cos\phi \tag{9}$$

is a S. F., zero over $AK'B$, where $r=c$, $\Omega=\Omega'$, and over the bowl AKB,

$$M_1-M_2=4\pi\sin^2\phi=4\pi\frac{y^2}{c^2} \tag{10}$$

Thus M_1-M_2 is the S. F. of the liquid motion at the instant when the spherical lid AKB is dropped on the spherical bowl $AK'B$, or lifted again, the liquid velocity being great where the windage is small.

An interpretation can also be given to the motion where the V. F. is

$$\frac{M_1-M_2}{r\sin\phi}\cos\psi=\left[\left(\frac{r}{c^2}\,\Omega-\frac{c}{r^2}\,\Omega'\right)\sin\phi+\left(\frac{r}{c^2}-\frac{1}{r}\right)Q\cos\phi\right]\cos\psi, \quad (11)$$

the lid AKB sliding across the bowl $AK'B$.

20. Figures of revolution can also be made in fig. 1 about the axis Oy. For a uniform rod AB, or a confocal prolate spheroid with an electric charge E,

$$V=\frac{2E}{AB}\,\mathrm{th}^{-1}\frac{AB}{PA+PB}, \quad N=E(PA-PB), \quad (1)$$

obtained from the integral for the potential of a uniform rod

$$\int_{-a}^{a}\frac{dy'}{\sqrt{[x^2+(y'-y)^2]}}=\mathrm{sh}^{-1}\frac{y+a}{x}-\mathrm{sh}^{-1}\frac{y-a}{x}=\mathrm{ch}^{-1}\frac{r_1}{x}-\mathrm{ch}^{-1}\frac{r_2}{x}$$

$$=\mathrm{sh}^{-1}\frac{a(r_1+r_2)-y(r_1-r_2)}{x^2}=\mathrm{ch}^{-1}\frac{r_1r_2-y^2+a^2}{x^2}$$

$$=2\mathrm{th}^{-1}\sqrt{\frac{r_1r_2-x^2-y^2+a^2}{r_1r_2+x^2-y^2+a^2}}=2\mathrm{th}^{-1}\frac{2a}{r_1+r_2}=2\mathrm{th}^{-1}\frac{2y}{r_1-r_2}. \quad (2)$$

Then

$$V=\mathrm{sh}^{-1}\frac{y+a}{x}=\mathrm{ch}^{-1}\frac{r_1}{x}, \quad N=r_1, \quad (3)$$

gives the P. F. and S. F. for a positive semi-infinite rod Ay, and negative rod Ay'. With the break at O, and $a=0$,

$$V=\mathrm{sh}^{-1}\frac{y}{x}=\mathrm{sh}^{-1}\tan\theta=\mathrm{ch}^{-1}\sec\theta=\log(\sec\theta+\tan\theta), \quad N=r, \quad (4)$$

so that the stream sheets are spherical and of uniform thickness, the flow issuing from a pole, and disappearing at the other pole, with velocity inversely as the distance from the axis.

In the conformal representation on a Mercator chart, there would be a uniform current, North and South. The S. F.

$$N=E(PA+PB)\text{ has the P. F. }\frac{2E}{AB}\,\mathrm{th}^{-1}\frac{2y}{r_1+r_2}=\frac{2E}{AB}\,\mathrm{th}^{-1}\frac{PA-PB}{AB}, \quad (5)$$

as required for confocal hyperboloids of revolution about the focal line AB, and excentricity $AB\div(PA-PB)$.

The typical element of a P. F. is $\frac{1}{r}$, as of a point source or sphere; and then the S. F. is $\cos\theta$ or $\sin\theta$ according as Ox or Oy is the axis.

But the simplest element of a S. F. is r, as in (4), for a line source Oy, and a line sink Oy'.

The single line source Oy would have the

$$\text{S. F. } N=r-y, \text{ and V. F. } V=\operatorname{sh}^{-1}\frac{y}{x}+\log x=\log(r+y), \tag{6}$$

and surfaces of constant N and V are orthogonal confocal paraboloids, with focus at O.

21. For a magnetic molecule, or sphere magnetized uniformly or moving in infinite liquid in direction Ox,

$$V=-\frac{d}{dx}\frac{1}{r}=\frac{x}{r^3},\quad N=\frac{d}{dx}\cos\phi=-\sin\phi\frac{d\phi}{dx}=\frac{\sin^2\phi}{r}=\frac{y^2}{r^3}, \tag{1}$$

$$\frac{dN}{dx}=-y\frac{dV}{dy}=-\frac{3xy^2}{r^5},\quad \frac{dN}{dy}=y\frac{dV}{dx}=\frac{2y}{r^3}-\frac{3xy^2}{r^5}. \tag{2}$$

For a magnetized oblate spheroid, on a focal circle AB, magnetized axially, or for the liquid motion due to axial velocity U,

$$V=-\frac{UxA(\lambda)}{2B(0)},\qquad N=\frac{\frac{1}{2}Uy^2B(\lambda)}{B(0)}, \tag{3}$$

$$A(\lambda)=\tan\omega-\omega,\quad B(\lambda)=\tfrac{1}{2}\omega-\tfrac{1}{4}\sin 2\omega,\quad \sin\omega=\frac{AB}{PA+PB}, \tag{4}$$

and $A(0)$, $B(0)$ the value of $A(\lambda)$, $B(\lambda)$ over the surface of the spheroid.

For a prolate spheroid with a focal line AB, magnetized or moving axially,

$$V=-\frac{UyA(\lambda)}{2B(0)},\qquad N=\frac{\frac{1}{2}Uy^2B(\lambda)}{B(0)}, \tag{5}$$

$$A(\lambda)=\tfrac{1}{4}\operatorname{sh}2\zeta-\tfrac{1}{2}\zeta,\quad B(\lambda)=\zeta-\operatorname{th}\zeta,\quad \operatorname{th}\zeta=\sin=\frac{AB}{PA+PB}=\frac{2a}{r_1+r_2}, \tag{6}$$

and then

$$B(\lambda)=\operatorname{th}^{-1}\frac{2a}{r_1+r_2}-\frac{2a}{r_1+r_2},$$

$$A(\lambda)=\tfrac{1}{4}\left(\frac{2}{\dfrac{r_1+r_2}{2a}-\dfrac{2a}{r_1+r_2}}-\operatorname{sh}^{-1}\frac{2}{\dfrac{r_1+r_2}{2a}-\dfrac{2a}{r_1+r_2}}\right) \tag{7}$$

22. The expression for the potential W of the solid homogeneous ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1 \tag{1}$$

can be derived from the components of the attraction, in the manner of §§ 8, 10,

A. J. M., p. 387, for the lens, by treating the potential as a homogeneous function of the second degree in x, y, z, and α, β, or γ, so that

$$2W=\alpha\frac{dW}{d\alpha}+x\frac{dW}{dx}+y\frac{dW}{dy}+z\frac{dW}{dz}. \tag{2}$$

It is not so difficult then (Thomson and Tait, § 494), to prove, as in Dirichlet's manner that the components of the attraction are given by

$$\frac{dW}{dx},\ \frac{dW}{dy},\ \frac{dW}{dz}=-\int_\lambda^\infty \frac{4\pi G\rho\alpha\beta\gamma\,(x,\,y,\,z)}{\psi+\alpha^2,\ \psi+\beta^2,\ \psi+\gamma^2}\,\frac{d\psi}{P(\psi)}, \tag{3}$$

$$P^2(\psi)=4\cdot\psi+\alpha^2\cdot\psi+\beta^2\cdot\psi+\gamma^2, \tag{4}$$

and the confocal ellipsoid through the attracted point (x, y, z) is

$$\frac{x^2}{\lambda+\alpha^2}+\frac{y^2}{\lambda+\beta^2}+\frac{z^2}{\lambda+\gamma^2}=1; \tag{5}$$

while $\alpha\dfrac{dW}{d\alpha}$, due to the uniform swelling of the ellipsoid in (1), whereby $(\alpha, \beta, \gamma, p)$ increases slightly by $k(\alpha, \beta, \gamma, p)$, p denoting the perpendicular from the centre on a tangent plane.

This makes dW the P. F. of a film, or coat of matter like paint, of thickness kp and superficial density $\sigma=k\rho p$, equivalent of an electrical film of superficial electrical density $kG\rho p$, and charge

$E=kG\rho\int pdS=kG\rho$ (three times the volume of the ellipsoid, or $4\pi\alpha\beta\gamma$),

and then $\sigma=\dfrac{Ep}{4\pi G\alpha\beta\gamma}$. (6)

Assuming the expression for the potential of this electrical distribution in equilibrium as

$$\int_\lambda^\infty \frac{Ed\psi}{P(\psi)}, \tag{7}$$

this makes, with $d\alpha$ for $k\alpha$,

$$dW=\int\frac{4\pi kG\rho\alpha\beta\gamma d\psi}{P(\psi)},\quad \alpha\frac{dW}{d\alpha}=\int\frac{4\pi G\rho\alpha\beta\gamma d\psi}{P(\psi)} \tag{8}$$

Thence the complete expression of W as

$$W=\int_\lambda^\infty\left(1-\frac{x^2}{\psi+\alpha^2}-\frac{y^2}{\psi+\beta^2}-\frac{z^2}{\psi+\gamma^2}\right)\frac{2\pi G\rho\alpha\beta\gamma d\psi}{P(\psi)}, \tag{9}$$

in which each term has received a physical interpretation.

If (x, y, z) is inside the ellipsoid, the lower limit λ must be replaced by zero.

23. In Professor Andrew Gray's method, "On the Attraction of a Spherical and Ellipsoidal Shell" (*Proceedings Edinburgh Mathematical*

Society, 1914), he produces CE in figure 3 to meet the concentric sphere through P in E', and then AE' is parallel to $P'E$ and equal to EP,

$$\frac{dF}{G\sigma}=\frac{dS\cos\theta}{Ep^2}=\frac{CA^2}{CP^2}\frac{dS'\cos\theta}{AE'^2}=\frac{c^2}{r^2}d\Omega', \tag{1}$$

where $d\Omega'$ is the conical angle subtended at A by dS' at E'.

He extends the method to the ellipsoidal shell through E in figure 4, drawing the confocal ellipsoid through P, and taking A, P, and E, E' corresponding points on the ellipsoids, so that $AE'=EP$.

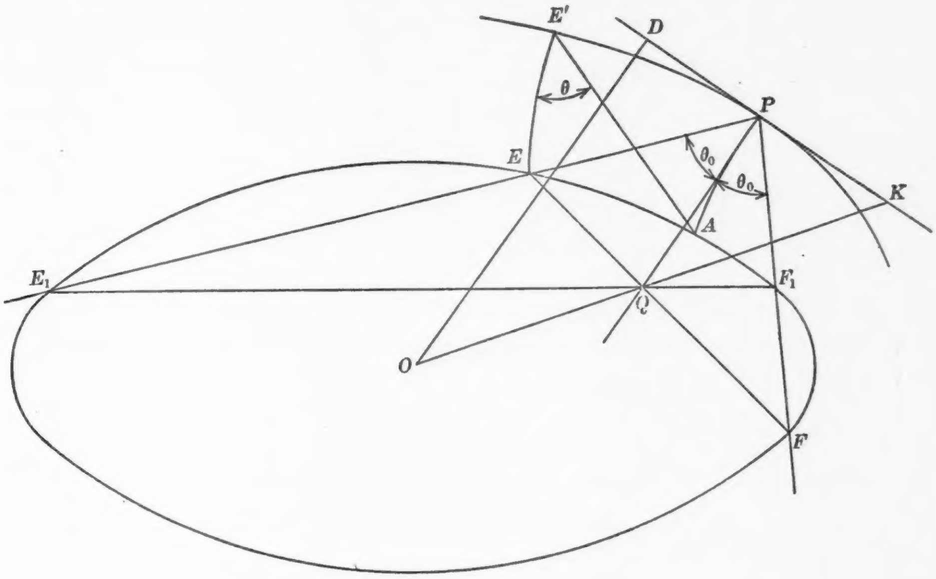

FIG. 4.

Draw the diameter parallel to AE, cutting the tangent plane at A, E in W_0, W; and denote the perpendicular from the centre O on the tangent plane at A, E by p_0, p; and denote the angle which AE makes with the normal at A, E by ϕ_0, ϕ; then $p_0 \sec\phi_0=OW_0$, $p\sec\phi=OW$; and the plane through O and the line of intersection of the tangent planes at A, E bisects AE, so that

$$OW_0=OW, \quad p_0\sec\phi_0=p\sec\phi. \tag{2}$$

Similarly with ϖ_0, ϖ the perpendicular from the centre on the tangent planes at P, E', and θ_0, θ the angles which EP, AE' make with the normal at P, E', Professor Gray finds

$$\varpi_0\sec\theta_0=\varpi\sec\theta; \tag{3}$$

the equivalent of $OV_0=OV$, if OV_0, OV, parallel to EP, AE' cut the tangent planes at P, E' in V_0, V.

For if (x, y, z), (x', y', z') are the coordinates of P, E,

$$\frac{PE}{OV_0}=\frac{\text{perpendicular from } E \text{ on the tangent plane at } P}{\text{perpendicular from } O \text{ on the tangent plane at } P}$$

$$=1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}, \tag{4}$$

and this ratio is unaltered in $\dfrac{AE'}{OV}$, because the coordinates of the corresponding points A, E' are

$$x''=x\sqrt{\frac{\alpha^2}{\lambda+\alpha^2}},\quad y''=y\sqrt{\frac{\beta^2}{\lambda+\beta^2}},\quad z''=z\sqrt{\frac{\gamma^2}{\lambda+\gamma^2}}, \tag{5}$$

$$x'''=x'\sqrt{\frac{\lambda+\alpha^2}{\alpha^2}},\quad y'''=y'\sqrt{\frac{\lambda+\beta^2}{\beta^2}},\quad z'''=z'\sqrt{\frac{\lambda+\gamma^2}{\gamma^2}}, \tag{6}$$

so that

$$x''x'''=xx',\quad y''y'''=yy',\quad z''z'''=zz', \tag{7}$$

$$\frac{AE'}{OV}=1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}=\frac{PE}{OV_0} \tag{8}$$

But from the fundamental property of corresponding points

$$\begin{aligned}AE'^2&=\left(x\sqrt{\frac{\alpha^2}{\lambda+\alpha^2}}-x'\sqrt{\frac{\lambda+\alpha^2}{\alpha^2}}\right)^2+\dots\\
&=x^2\frac{\alpha^2}{\lambda+\alpha^2}-2xx'+x'^2\frac{\lambda+\alpha^2}{\alpha^2}+\dots\\
&=x^2\left(1-\frac{\lambda}{\lambda+\alpha^2}\right)-2xx'+x'^2\left(\frac{\lambda}{\alpha^2}+1\right)+\dots\\
&=x^2+y^2+z^2-\lambda-2xx'-2yy'-2zz'+\lambda+x'^2+y'^2+z'^2\\
&=(x-x')^2+(y-y')^2+(z-z')^2=PE^2.\end{aligned} \tag{9}$$

$$OV_0=OV,\quad \varpi_0\sec\theta_0=\varpi\sec\theta. \tag{10}$$

Draw a cone of small conical angle $d\Omega$ from the vertex at P to cut out area elements dS, dS_1 of the ellipsoid at E, E_1, the axis PEE_1 at an angle ϕ, ϕ_1 with the normal at E, E_1, so that

$$dS=PE^2\sec\phi d\Omega,\quad dS_1=PE_1^2\sec\phi_1 d\Omega. \tag{11}$$

Then with superficial density on the ellipsoid, proportioned to p the perpendicular on the tangent plane, the ratio of the attraction of these elements at E, E_1 on P is

$$\frac{\dfrac{pdS}{PE^2}}{\dfrac{p_1dS_1}{PE_1^2}}=\frac{p\sec\phi}{p_1\sec\phi_1}=1. \tag{12}$$

Thus, if P was inside the ellipsoid, the attraction of the elements would be equal and opposite, and the attraction of the shell is zero.

But with P external the normal PQ of the confocal through P is the axis of the enveloping cone; another elementary cone PF_1F can be drawn equally inclined to PQ, of the same small conical angle $d\Omega$, and contributing an equal attraction; also EF, E_1F_1 pass through Q, and make equal angles with PQ, where Q is the pole of the tangent plane at P with respect to the ellipsoid through E.

The surface of the ellipsoid can be exhausted by such pairs of cones of equal attraction, equally inclined to the normal PQ, so that the resultant attraction at P of the ellipsoidal shell is along the normal of the confocal through P, which confocal is thus a level surface.

Moreover, if the shell is divided by the plane of contact of the tangent cone from P, the two parts of the shell exert equal attraction on P.

24. With dS' the element of surface at E' on the ellipsoid PE', corresponding to dS at E on AE,

$$\frac{\varpi dS'}{pdS} = \text{ratio of corresponding conical elements of volume with vertex at } O$$

$$= \text{ratio of whole volume of ellipsoids} = \sqrt{\frac{\lambda+\alpha^2\cdot\lambda+\beta^2\cdot\lambda+\gamma^2}{\alpha^2\beta^2\gamma^2}} \qquad (1)$$

and then with superficial density $k\rho p$ at E, the normal component dF of the attraction at P of the element dS at E is given by

$$\frac{dF}{kG\rho} = \frac{p\cos\theta_0 dS}{PE^2} = \frac{\varpi\cos\theta_0 dS'}{PE^2}\sqrt{\frac{\alpha\beta\gamma}{(\lambda+\alpha^2\cdot\lambda+\beta^2\cdot\lambda+\gamma^2)}}$$

$$= \frac{\varpi_0\cos\theta dS'}{AE'^2}\frac{2\alpha\beta\gamma}{P(\lambda)} = \varpi_0 d\Omega'\frac{2\alpha\beta\gamma}{P(\lambda)}. \qquad (2)$$

Thus, for the whole ellipsoid, $\Omega'=4\pi$, as A is inside the ellipsoid PE', and

$$\frac{F}{kG\rho} = 4\pi\omega_0\frac{2\alpha\beta\gamma}{P(\lambda)} = 4\pi\varpi_0\frac{\text{volume of ellipsoid } AE}{\text{volume of ellipsoid } PE'}. \qquad (3)$$

The potential U of this shell is then the work required to carry P off to infinity, and so

$$U=\int F d\varpi_0, \text{ with } \varpi_0 d\varpi_0=\tfrac{1}{2}d\lambda. \qquad (4)$$

The electric charge E, or mass of the shell, is given by

$$E=\int kG\rho p dS=4\pi kG\rho\alpha\beta\gamma, \qquad (5)$$

$$U=\int_\lambda^\infty\frac{4\pi kG\rho\alpha\beta\gamma d\lambda}{P(\lambda)}=\int_\lambda^\infty\frac{Ed\lambda}{P(\lambda)} \qquad (6)$$

and the level surfaces are the confocal ellipsoids of the system

$$\frac{x^2}{\lambda+\alpha^2}+\frac{y^2}{\lambda+\beta^2}+\frac{z^2}{\lambda+\gamma^2}=1. \tag{7}$$

And conversely, U in (6) is the potential of a mass E, if its external level surfaces are given by the system in (7).

This is Professor Gray's proof; but our desideratum is to refer Ω' to some internal point P', analogous in the sphere of figure 3 to the inverse point of P.

The pole of the tangent plane at P with respect to the ellipsoid AE will be at Q on the normal PQ, with coordinates

$$\frac{\alpha^2 x}{\lambda+\alpha^2},\quad \frac{\beta^2 y}{\lambda+\beta^2},\quad \frac{\gamma^2 z}{\lambda+\gamma^2}; \tag{8}$$

and PQ will be the normal at Q of the ellipsoid through Q, homothetic with the ellipsoid through P.

If EQ makes an angle θ with the normal at E,

$$\cos\theta=\frac{px'}{a^2}\,\frac{x'-\dfrac{\alpha^2 x}{\lambda+\alpha^2}}{EQ}+\ldots, \tag{9}$$

$$EQ\cos\theta=p\left(1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}\right), \tag{10}$$

$$EP\cos\theta_0=\varpi_0\left(1-\frac{xx'}{\lambda+\alpha^2}-\frac{yy'}{\lambda+\beta^2}-\frac{zz'}{\lambda+\gamma^2}\right), \tag{11}$$

$$\frac{EQ}{EP}=\frac{p\sec\theta}{\varpi_0\sec\theta_0}, \tag{12}$$

$$\frac{dF}{kG\rho}=\frac{p\cos\theta_0 dS}{EP^2}=\frac{EQ}{EP^3}\varpi_0\cos\theta dS=\frac{EQ^3}{EP^3}\varpi_0 d\Omega'. \tag{13}$$

But this is not a constant multiple of $d\Omega'$ on the ellipsoid, but only on the sphere, and so the analogy breaks down with P' at Q, and some other position must be found for P', say on the line of force PA.

25. In a reduction to a standard form of the elliptic integral, take

$$\alpha^2<\beta^2<\gamma^2, \text{ and then } s_1>s_2>s_3, \tag{1}$$

on putting

$$\psi+\alpha^2,\ \psi+\beta^2,\ \psi+\gamma^2=m^2(s-s_1,\ s-s_2,\ s-s_3), \tag{2}$$

$$\frac{d\psi}{P(\psi)}=\frac{ds}{m\sqrt{S}},\quad \int_\lambda^\infty\frac{\sqrt{(\gamma^2-\alpha^2)}\,d\psi}{P(\psi)}=\int_\sigma^\infty\frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}}=eK, \tag{3}$$

$$\text{sn}^2eK=\frac{s_1-s_3}{\sigma-s_3}=\frac{\gamma^2-\alpha^2}{\lambda+\gamma^2},\quad \text{cn}^2eK=\frac{\lambda+\alpha^2}{\lambda+\gamma^2},\quad \text{dn}^2eK=\frac{\lambda+\beta^2}{\lambda+\gamma^2} \tag{4}$$

$$\kappa^2=\frac{\gamma^2-\beta^2}{\gamma^2-\alpha^2},\quad \kappa'^2=\frac{\beta^2-\alpha^2}{\gamma^2-\alpha^2}. \tag{5}$$

Then, putting $\lambda=0$, $E=1$ in (6), § 24, the reciprocal of the capacity of the ellipsoid (1), § 22, is

$$\int_0^\infty \frac{d\psi}{P(\psi)}=\frac{eK}{\sqrt{(\gamma^2-\alpha^2)}}=\frac{\operatorname{cn}^{-1}\dfrac{\alpha}{\gamma}}{\sqrt{(\gamma^2-\alpha^2)}}. \tag{6}$$

Thus, to determine λ for an ellipsoid of double capacity, we take the formulas for $\frac{1}{2}eK$,

$$\operatorname{sn}^2\tfrac{1}{2}eK=\frac{1-\operatorname{cn}eK}{1+\operatorname{dn}eK},\quad \frac{\gamma^2-\alpha^2}{\lambda+\gamma^2}=\frac{\gamma-\alpha}{\gamma+\beta}, \tag{7}$$

$$\lambda+\gamma^2=\gamma+\alpha\cdot\gamma+\beta,\quad \lambda+\beta^2=\beta+\gamma\cdot\beta+\alpha,\quad \lambda+\alpha^2=\alpha+\beta\cdot\alpha+\gamma, \tag{8}$$

(Hargreave's *Messenger of Mathematics*, August, 1912).

For an oblate ellipsoid, $\beta=\gamma=\alpha\sec\theta$, $\kappa=0$,

$$\int_0^\infty \frac{d\psi}{P(\psi)}=\int\frac{d\psi}{2(\psi+\beta^2)\sqrt{(\psi+\alpha^2)}}=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\cos^{-1}\frac{\alpha}{\gamma}$$

$$=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\sin^{-1}\frac{\sqrt{(\gamma^2-\alpha^2)}}{\gamma}=\frac{\theta}{\gamma\sin\theta}=\frac{\theta}{\alpha\tan\theta}. \tag{9}$$

For a prolate ellipsoid, $\alpha=\beta$, $\kappa=1$,

$$\int_0^\infty \frac{d\psi}{P(\psi)}=\int\frac{d\psi}{2(\psi+\alpha^2)\sqrt{(\psi+\gamma^2)}}=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\operatorname{ch}^{-1}\frac{\gamma}{\alpha}$$

$$=\frac{1}{\sqrt{(\gamma^2-\alpha^2)}}\operatorname{sh}^{-1}\frac{\sqrt{(\gamma^2-\alpha^2)}}{\alpha}; \tag{10}$$

giving the capacity of the sphere by the radius, when $\alpha=\beta=\gamma$.

For an elliptic plate $\alpha=0$, $\operatorname{cn}^{-1}0=K$, $\kappa'=\dfrac{\beta}{\gamma}$, making the capacity $\gamma\div K$; agreeing with the circular plate when $\beta=\gamma$, $\kappa=0$, $K=\frac{1}{2}\pi$. Or, as in § 8, the capacity of the elliptic plate is the Gauss A. G. M. of β and γ, divided by $\frac{1}{2}\pi$.

To determine λ for an ellipsoid of n-fold capacity of the elliptic plate requires the elliptic functions of K/n.

Thus, as before in (8), for double capacity, $\lambda=\beta\gamma$ with $\alpha=0$.

For three-fold capacity, from the formula

$$\operatorname{sn}\tfrac{1}{3}K+\operatorname{cn}\tfrac{2}{3}K=1,\quad \operatorname{sn}\tfrac{1}{3}K+\frac{\kappa'\operatorname{sn}\frac{1}{3}K}{\operatorname{dn}\frac{1}{3}K}=1, \tag{11}$$

$$\frac{\sqrt{(\lambda+\gamma^2)}}{\gamma}+\frac{\beta\sqrt{(\lambda+\gamma^2)}}{\gamma\sqrt{(\lambda+\beta^2)}}=1,\quad \frac{\gamma}{\sqrt{(\lambda+\gamma^2)}}-\frac{\beta}{\sqrt{(\lambda+\beta^2)}}=1, \tag{12}$$

leading on rationalisation to

$$\lambda^4-6\beta^2\gamma^2\lambda^2-4(\beta^2+\gamma^2)\beta^2\gamma^2\lambda-3\beta^4\gamma^4=0, \tag{13}$$

a Jacobian quartic for λ.

Or, for a condenser of capacity 50 per cent greater than the plate,

$$\frac{\mathrm{cn}\,\tfrac{2}{3}K}{\mathrm{dn}\,\tfrac{2}{3}K}+\mathrm{cn}\tfrac{2}{3}K=1,\quad \sqrt{\frac{\lambda}{\lambda+\beta^2}}+\sqrt{\frac{\lambda}{\lambda+\gamma^2}}=1,\quad \frac{1}{\sqrt{(\lambda+\beta^2)}}+\frac{1}{\sqrt{(\lambda+\gamma^2)}}=\frac{1}{\sqrt{\lambda}},\tag{14}$$

$$3\lambda^4+4(\beta^2+\gamma^2)\lambda^3+6\beta^2\gamma^2\lambda^2-\beta^4\gamma^4=0,\tag{15}$$

the same Jacobian quartic (13) as before, with the substitution $\left(\lambda, \frac{\beta\gamma}{\lambda}\right)$.

26. With (9), § 22, for the homogeneous ellipsoid of mass M,

$$\frac{W}{\tfrac{3}{2}GM}=D(\lambda)-x^2A(\lambda)-y^2B(\lambda)-z^2C(\lambda),\tag{1}$$

changing in the interior to

$$\frac{W}{\tfrac{3}{2}GM}=D(0)-x^2A(0)-y^2B(0)-z^2C(0),\tag{2}$$

$$D(\lambda)=\int_\lambda^\infty\frac{d\psi}{P(\psi)}=\frac{eK}{\sqrt{(\gamma^2-\alpha^2)}},\quad \mathrm{sn}^2eK=\frac{\gamma^2-\alpha^2}{\lambda+\gamma^2},\quad \kappa^2=\frac{\gamma^2-\beta^2}{\gamma^2-\alpha^2},\tag{3}$$

$$A(\lambda),\,B(\lambda),\,C(\lambda)=\int_\lambda^\infty\frac{1}{\psi+\alpha^2,\ \psi+\beta^2,\ \psi+\gamma^2}\,\frac{d\psi}{P(\psi)}=-\frac{2dD(\lambda)}{d(\alpha^2,\beta^2,\gamma^2)},\tag{4}$$

$$A(\lambda)+B(\lambda)+C(\lambda)=\frac{2}{P(\lambda)},\tag{5}$$

and then with

$$\frac{1}{\psi+\alpha^2},\ \frac{1}{\psi+\beta^2},\ \frac{1}{\psi+\gamma^2}=\frac{1}{\gamma^2-\alpha^2}\left(\frac{\mathrm{sn}^2u}{\mathrm{cn}^2u},\ \frac{\mathrm{sn}^2u}{\mathrm{dn}^2u},\ \mathrm{sn}^2u\right)\tag{6}$$

$$A(\lambda),\,B(\lambda),\,C(\lambda)=\frac{1}{(\gamma^2-\alpha^2)^{\frac{3}{2}}}\int_0^{eK}\left(\frac{\mathrm{sn}^2u}{\mathrm{cn}^2u},\ \frac{\mathrm{sn}^2u}{\mathrm{dn}^2u},\ \mathrm{sn}^2u\right)du,\tag{7}$$

three Elliptic Integrals of the Second Kind,

$$(\gamma^2-\alpha^2)^{\frac{3}{2}}[A(\lambda),\,B(\lambda),\,C(\lambda)]$$
$$=\frac{\mathrm{zs}\,(1-e)K-eE}{\kappa'^2},\quad \frac{e(E-\kappa'^2K)-\mathrm{zn}\,(1-e)K}{\kappa^2\kappa'^2},\quad \frac{e(K-E)-\mathrm{zn}eK}{\kappa^2}.\tag{8}$$

27. An ocean film, covering the surface of a Jacobian rotating ellipsoid, would have the depth inversely as the local gravity, or directly as p the perpendicular on the tangent plane, neglecting its self-gravitation; and so it would resemble the electrical coating or charge on the insulated ellipsoid.

But on Maclaurin's theorem, if the matter of the ellipsoid, § 22, was cut out and condensed on the surface, this film would have the same exterior potential as the solid ellipsoid if the thickness was inversely as p, as if a thin focoid of uniform density.

Maclaurin's theorem is equivalent then to saying that the external potential of the solid ellipsoid is unaltered if a confocal cavity is excavated, and the stuff condensed uniformly in the remaining focoidal shell.

Denoting by l, m, n, the direction cosines of the perpendicular p and p_1 on parallel tangent planes of the confocal ellipsoids defined by λ and λ_1,

$$p^2=(\lambda+\alpha^2)l^2+(\lambda+\beta^2)m^2+(\lambda+\gamma^2)n^2=\lambda+\alpha^2l^2+\beta^2m^2+\gamma^2n^2,$$
$$p_1^2=\lambda_1+\alpha^2l^2+\beta^2m^2+\gamma^2n^2; \tag{1}$$

and defining the thickness of the focoidal shell by the distance between the two parallel tangent planes

$$p_1^2-p^2=\lambda_1-\lambda,\quad p_1-p=\frac{\lambda_1-\lambda}{p_1+p}\ ; \tag{2}$$

reducing for a thin shell to a thickness

$$dp=\frac{d\lambda}{2p}. \tag{3}$$

In Thomson and Tait's "Natural Philosophy" (T and T', § 494 d,) the problem is considered in the reverse way by taking a P. F.

$$V=\tfrac{1}{2}\left(1-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}-\frac{z^2}{\gamma^2}\right),\text{ inside the ellipsoid; } V=0 \text{ in exterior space;} \tag{4}$$

so that V is continuous, but its variation is discontinuous in crossing the ellipsoid; thence the appropriate density ρ is determined by Laplace's equation,

$$4\pi G\rho=-\nabla^2V=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2},\text{ inside the ellipsoid, but outside}=0; \tag{5}$$

and, in crossing the surface, the superficial density σ is given by

$$4\pi G\sigma=\frac{dV}{dp}=\frac{dV}{dx}\frac{px}{\alpha^2}+\ldots.=-\frac{px^2}{\alpha^4}\ldots.=-\frac{1}{p}. \tag{6}$$

Thus the focoidal film or shell of superficial density $\sigma=\dfrac{1}{4\pi Gp}$ will have the same exterior potential as the solid ellipsoid of density

$$\rho=\frac{1}{4\pi G}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right),$$

and the same mass; since, integrating over the surface S, and volume V,

$$\begin{aligned}\int\frac{dS}{4\pi Gp}&=\frac{1}{4\pi G}\int\frac{ldydz+mdzdx+ndxdy}{p}\\&=\frac{1}{4\pi G}\int\left(\frac{xdydz}{\alpha^2}+\frac{ydzdx}{\beta^2}+\frac{zdxdy}{\gamma^2}\right)\\&=\frac{V}{4\pi G}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)=V\rho=M.\end{aligned} \tag{7}$$

Thence for the focoid shell

$$\text{outside, } W = \tfrac{3}{2} GM \int_\lambda^\infty \left(1 - \frac{x^2}{\psi+\alpha^2} - \frac{y^2}{\psi+\rho^2} - \frac{z^2}{\psi+\gamma^2}\right) \frac{d\psi}{P(\psi)} \tag{8}$$

$$\text{inside, } W = \tfrac{3}{2} GM \int_0^\infty \left(1 - \frac{x^2}{\psi+\alpha^2} - \frac{y^2}{\psi+\rho^2} - \frac{z^2}{\psi+\gamma^2}\right) \frac{d\psi}{P(\psi)}$$

$$- \tfrac{3}{2} GM \frac{\alpha\beta\gamma}{\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2} \left(1 - \frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2}\right). \tag{9}$$

So that if the solid ellipsoid (1), § 22, is not homogeneous, but stratified in confocals, the exterior potential is unaltered.

But if the strata are similar homothetic ellipsoids, then

$$W = 2\pi G\alpha\beta\gamma \int_{\lambda,\,0}^\infty f\left(1 - \frac{x^2}{\psi+\alpha^2} - \frac{y^2}{\psi+\beta^2} - \frac{z^2}{\psi+\gamma^2}\right) \frac{d\psi}{P(\psi)}, \tag{10}$$

where the density is given by

$$\rho = f'\left(1 - \frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2}\right). \tag{11}$$

Thus, if there is a cavity of semi-axes $k(\alpha, \beta, \gamma)$ in a thick homeoid, the potential in the interior is given by

$$W = 2\pi G\alpha\beta\gamma [f(1-k^2) - f(0)] \int_0^\infty \frac{d\psi}{P(\psi)}. \tag{12}$$

Consult memoirs on this subject by Ferrers and Dyson, in the *Quarterly Journal of Mathematics*, XIV, p. 1, 1877, and XXV, p. 259, 1890.

28. We resume here the interpretation of the terms in the potential W in (9), § 22, as we think it would strike a Maxwell or Clifford, in an investigation of the physical meaning of them, to show reason why these terms depending on the matter inside the ellipsoid and on the surface should arise in an integration extending through exterior space, beyond the attracted point away to infinity.

In his "Biographical Introduction" to Clifford's "Lectures and Essays," Sir Frederick Pollock cites his reminiscence of a walk with Clifford at Cambridge, "when Clifford explained to him in words the inner meaning of Ivory's theorem, and its geometrical conditions, omitting all the formidable apparatus of coordinates and integrals, where the chain of symbolic proof seemed artificial and dead, and failed to satisfy the reason where it compelled the understanding."

Clifford's line of argument is probably the same as that incorporated later in T and T', § 532, depending on Ivory's theorem of corresponding points.

In an associated problem of hydrodynamics or induced magnetism, we can take the function $\frac{dW}{dx}$ as a new P. F., and give a physical interpretation.

According to Maxwell (E. and M. II, § 437), $\frac{dW}{dx}$ for any attracting solid will give the magnetic potential or velocity function of the body, only however, when $\frac{dW}{dx}$ is a linear function of the coordinates x, y, z, within the body, and W is then a quadratic function in the interior.

The only case with which we are acquainted in which W is a quadratic function of the coordinates within the body is that in which the body is bounded by a complete surface of the second degree.

But in the case of a lens, bounded by portions of a spherical surface, W is not a quadratic function, so that $\frac{dW}{dx}$ will not give a velocity function for the movement of the surface, or a magnetic potential for uniform magnetization in that direction.

29. Begin by considering the hydrodynamical interpretation of the component $\frac{dW}{dx}$ of the attraction of the ellipsoid, by taking an equivalent velocity function

$$\phi = xA(\lambda), \text{ or more generally, } x[A(\lambda)-A(\lambda_1)] = \int_\lambda^{\lambda_1} \frac{x}{\psi+\alpha^2}\frac{d\psi}{P(\psi)}. \qquad (1)$$

The up-gradient of ϕ in the direction of the normal of the confocal λ is then, writing A, A_1 for $A(\lambda)$, $A(\lambda_1)$,

$$\frac{d\phi}{dp} = \frac{dx}{dp}(A-A_1) + x\frac{dA}{dp} = l(A-A_1) + 2px\frac{dA}{d\lambda}$$
$$= l\left[A-A_1+2(\lambda+\alpha^2)\frac{dA}{d\lambda}\right] = 2l\sqrt{(\lambda+\alpha^2)}\frac{d}{d\lambda}[\sqrt{(\lambda+\alpha^2)}(A-A_1)], \qquad (2)$$

or with

$$\frac{dA}{d\lambda} = -\frac{1}{\lambda+\alpha^2}\frac{1}{P(\lambda)}, \quad 2(\lambda+\alpha^2)\frac{dA}{d\lambda} = -\frac{2}{P(\lambda)} = -A-B-C, \qquad (3)$$

$$\frac{d\phi}{dp} = -l(A_1+B+C) = -lu, \quad u = A_1+B+C, \qquad (4)$$

taking the down-gradient of ϕ as the velocity; and this shows that any confocal λ may be supposed to swim in the liquid for a moment, without distortion, with this velocity u parallel to Ox.

At infinity, B and $C=0$, and $u=A_1$, so that $A_1=0$, $\lambda_1=\infty$ is required to make the velocity there zero.

But in (4) the normal velocity is zero over the ellipsoid λ_2, where

$$B(\lambda_2)+C(\lambda_2)=-A(\lambda_1), \tag{5}$$

requiring $\lambda_1+\alpha^2$ to be negative, and making in (1),

$$\phi=x[A(\lambda)+B(\lambda_2)+C(\lambda_2)]. \tag{6}$$

Thus, if there is an infinite stream with velocity $-U$ parallel to Ox, and the ellipsoid λ_2 is inserted, the velocity function is changed from Ux to

$$\phi=Ux\frac{A(\lambda)+B(\lambda_2)+C(\lambda_2)}{B(\lambda_2)+C(\lambda_2)}=Ux\left(\frac{A}{B_2+C_2}+1\right); \tag{7}$$

and with the stream reduced to rest, and the ellipsoid λ_2 advancing with velocity U,

$$\phi=Ux\frac{A}{B_2+C_2}. \tag{8}$$

And generally, in the space between the ellipsoids λ_3, λ_4,

$$\phi=x\frac{u_3(A+B_4+C_4)-u_4(A+B_3+C_3)}{B_3+C_3-B_4-C_4}, \tag{9}$$

with the ellipsoid λ_3, λ_4 advancing with velocity u_3, u_4; reducing with $u_3=u_4=U$ to $\phi=-Ux$.

30. To show how the continuity of the liquid requires that $A(\lambda)$ should have the form given in (1), § 29, consider the flow across the annular section $K-K_2$ made by $x=0$ of the ellipsoid λ, and an interior ellipsoid λ_2, moving with velocity u and u_2 along Ox.

Then $uK-u_2K_2$ is the rate of increase of volume between the two half ellipsoids, and this must be made good by filling up of the flow of the liquid across the annulus, with velocity $-A+A_1$, since $x=0$ makes $\frac{d\phi}{dx}=A-A_1$.

The integral equation of continuity is then

$$uK-u_2K_2+\int_{\lambda_2}^{\lambda}(A-A_1)\,dK=0, \tag{1}$$

and differentiating with respect to λ for the differential equation of continuity

$$\frac{d}{d\lambda}uK+(A-A_1)\frac{dK}{d\lambda}=0,\quad \frac{d}{d\lambda}(u+A-A_1)K-\frac{dA}{d\lambda}K=0. \tag{2}$$

With the value of u from (2), (3), (4), § 29,

$$u+A-A_1=-2(\lambda+\alpha^2)\frac{dA}{d\lambda}, \tag{3}$$

$$2\frac{d}{d\lambda}\left[(\lambda+\alpha^2)\frac{dA}{d\lambda}K\right]+\frac{dA}{d\lambda}K=0,\quad 2(\lambda+\alpha^2)\frac{d}{d\lambda}\left(\frac{dA}{d\lambda}K\right)+3\frac{dA}{d\lambda}K=0, \tag{4}$$

and integrating
$$\frac{dA}{d\lambda}K(\lambda+\alpha^2)^{\frac{3}{2}}=\text{constant}; \tag{5}$$

so that with
$$K=\pi\sqrt{(\lambda+\beta^2\cdot\lambda+\gamma^2)}=\frac{\frac{1}{2}\pi P}{\sqrt{(\lambda+\alpha^2)}}, \tag{6}$$

$$\frac{dA}{d\lambda}=\frac{\text{constant}}{(\lambda+\alpha^2)P(\lambda)}, \tag{7}$$

as in (1), § 29.

31. When the ellipsoid of § 22 is of revolution, make the circle on AB the focal circle for an oblate spheroid by putting $\beta=\gamma$, as in (9), § 25, but for a prolate spheroid take AB as the focal line, and put $\beta=\alpha$, as in (10), § 25. Then in parallel columns:

Spheroid	Oblate, on axis Ox	Prolate, on axis Oy
$\lambda+\alpha^2$	$a^2\cot^2\omega=B^2$	$\frac{a^2}{\text{sh}^2\zeta}=B^2$
$\lambda+\beta^2$	$a^2\sec^2\omega$	$a^2\frac{\text{ch}^2\zeta}{\text{sh}^2\zeta}$
$\beta^2-\alpha^2$	a^2	a^2
$\frac{AB}{PA+PB}$	$\sin\omega$	$\text{th}\,\zeta$
$P(\lambda)$	$2a^3\frac{\cos\omega}{\sin^3\omega}$	$2a^3\frac{\text{ch}\,\zeta}{\text{sh}^3\zeta}$
$aD(\lambda)=\int_\lambda^\infty\frac{ad\psi}{P(\psi)}$	ω	ζ
$a^3A(\lambda)$	$\tan\omega-\omega$	$\frac{1}{4}\text{sh}2\zeta-\frac{1}{2}\zeta$
$a^3B(\lambda)$	$\frac{1}{2}\omega-\frac{1}{4}\sin 2\omega$	$\zeta-\text{th}\,\zeta$
$\frac{aW}{\frac{3}{2}GM}$	$\omega-\frac{x^2}{a^2}(\tan\omega-\omega)$ $-\frac{y^2+z^2}{a^2}(\frac{1}{2}\omega-\frac{1}{4}\sin 2\omega)$	$\zeta-\frac{x^2+z^2}{a^2}(\frac{1}{4}\text{sh}2\zeta-\frac{1}{2}\zeta)$ $-\frac{y^2}{a^2}(\zeta-\text{th}\,\zeta)$
At infinity	$\omega=\sin\omega=\frac{a}{r}$	$\zeta=\text{th}\,\zeta=\frac{a}{r}$
$a^3A(\infty)=a^3B(\infty)$	$\frac{1}{3}\omega^3$	$\frac{1}{3}\zeta^3$
$\frac{rW(\infty)}{\frac{3}{2}GM}$	$\frac{2}{3}$	$\frac{2}{3}$
$\frac{a^3N}{\frac{3}{2}GM}$	$xy^2(\frac{1}{2}\omega-\frac{1}{4}\sin 2\omega)+\frac{2}{3}x^3\tan^3\omega$	$x^2y(\frac{1}{4}\text{sh}^2\zeta-\frac{1}{2}\zeta)+\frac{2}{3}y^3\text{sh}^3\zeta$

The Hypergeometric Function in its Physical Applications.

32. Many of the functions required in a physical problem can be classed in a special case of the Hypergeometric (H. G.) Function, defined in Forsyth's "Differential Equations," Chapter VI.

Beginning with Euler's First and Second Integral, called the Gamma and Beta Function, defined by

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx=\frac{1}{n}\int_0^\infty e^{-\sqrt[n]{z}}dz, \tag{1}$$

$$B(m,n)=\int_0^1 s^{m-1}(1-s)^{n-1}ds=2\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta, \tag{2}$$

with $s=\sin^2\theta$, $1-s=\cos^2\theta$; they are connected by the relation

$$B(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}, \tag{3}$$

so that the Gamma Function alone requires tabulation, given by Bertrand (Integral Calculus) as $\log\Gamma n$, and between 0 and 1 for n, since $\Gamma(n+1)=n\Gamma n$.

Proceeding with a generalization, the definite integral

$$F(\alpha,\beta,\gamma,x)=A\int_0^1 s^{\beta-1}(1-s)^{\gamma-\beta-1}(1-xs)^{-\alpha}ds \tag{4}$$

defines the Hypergeometric Function, where A is chosen so as to make the function unity when $x=0$, and the function reduces to a Beta Function; so that

$$A\int_0^1 s^{\beta-1}(1-s)^{\gamma-\beta-1}ds=1,\quad \frac{1}{A}=B(\beta,\gamma-\beta). \tag{5}$$

Introducing the Elliptic Function, and putting

$$s=\operatorname{sn}^2 v,\quad 1-s=\operatorname{cn}^2 v,\quad 1-xs=\operatorname{dn}^2 v,\quad x=\kappa^2, \tag{6}$$

$$F(\alpha,\beta,\gamma,x)=2A\int_0^K(\operatorname{sn}v)^{2\beta-1}(\operatorname{cn}v)^{2\gamma-2\beta-1}(\operatorname{dn}v)^{-2\alpha+1}dv, \tag{7}$$

or with $v=eK$,

$$F(\alpha,\beta,\gamma,x)=2A\int_{e=0}^1(\operatorname{sn}eK)^{2\beta-1}(\operatorname{cn}eK)^{2\gamma-2\beta-1}(\operatorname{dn}eK)^{-2\alpha+1}K\,de, \tag{8}$$

where β and $\gamma-\beta$ must be positive; and this is in a form analogous to Euler's Second Integral in (2).

Then F is a solution of the hypergeometric differential equation (Forsyth's D. E., Chapter VI),

$$x(1-x)\frac{d^2F}{dx^2}+[\gamma-(\alpha+\beta+1)x]\frac{dF}{dx}-\alpha\beta F=0. \tag{9}$$

Schwartz has shown that $F(\alpha,\beta,\gamma,x)$ is an algebraical function of the fourth element x in the special cases identified by Klein in his "Ikosahedron" as the polyhedral functions.

The XXIV transformations of the H. G. function in (8) are then obtained from the four arguments:

$$v+0,\quad K,\quad K+K'i,\quad K'i, \tag{10}$$

with Abel's six linear transformations for the modulus:

$$x,\quad 1-x,\quad \frac{1}{1-x},\quad \frac{x}{x-1},\quad \frac{x-1}{x},\quad \frac{1}{x}, \tag{11}$$

a succession of the complementary and reciprocal modulus.

33. In the special case of $\alpha+\beta=1$, $\gamma=1$, then (9), § 32, becomes the D. E. of the zonal harmonic of order n, where

$$n(n+1)=-\alpha\beta=-N,\ \text{suppose},\ n+\tfrac{1}{2}=\sqrt{(\tfrac{1}{4}-N)}, \tag{1}$$

and for any order n, integral, fractional, or complex, the zonal harmonic of $x=\sin^2\frac{1}{2}\theta$ is

$$P_n=F=2A\int_0^K\left(\frac{\operatorname{sn}v\operatorname{dn}v}{\operatorname{cn}v}\right)^{\sqrt{(1-4N)}=2n+1}dv=A\int_0^{2K}\left(\frac{1-\operatorname{cn}w}{1+\operatorname{cn}w}\right)^{\sqrt{(\frac{1}{4}-N)}}dw, \tag{2}$$

with $w=2v$, and a modular angle $\frac{1}{2}\theta$.

Should $\frac{1}{4}-N$ be negative, a Mehler function arises of order $p=\sqrt{(N-\frac{1}{4})}$, given by

$$F=\int_0^{2K}\cos\log\left(\frac{1-\operatorname{cn}w}{1+\operatorname{cn}w}\right)^p dw, \tag{3}$$

satisfying the D. E.

$$x(1-x)\frac{d^2F}{dx^2}+(1-2x)\frac{dF}{dx}-(p^2+\tfrac{1}{4})F=0. \tag{4}$$

If we put

$$\frac{1+\operatorname{cn}w}{1-\operatorname{cn}w}=e^a,\quad \operatorname{cn}w=\operatorname{th}\tfrac{1}{2}a,\quad \operatorname{sn}w=\operatorname{sech}\tfrac{1}{2}a,\quad \operatorname{dn}w=\operatorname{sech}\tfrac{1}{2}a\sqrt{(\operatorname{ch}^2\tfrac{1}{2}a-x)}, \tag{5}$$

$$-\operatorname{sn}w\operatorname{dn}w\,dw=\tfrac{1}{2}\operatorname{sech}^2\tfrac{1}{2}a\,da,\quad dw=\frac{-\frac{1}{2}da}{\sqrt{(\operatorname{ch}^2\frac{1}{2}a-x)}}, \tag{6}$$

then with $p=\sqrt{(\frac{1}{4}-N)}$, as in (2),

$$F=A\int_{-\infty}^{\infty}e^{pa}\frac{\frac{1}{2}da}{\sqrt{(\operatorname{ch}^2\frac{1}{2}a-x)}}=A\int_{0,\,-\infty}^{\infty,\,0}\frac{e^{pa}da}{\sqrt{(\operatorname{ch}^2\frac{1}{2}a-x)}}=A\int_0^{\infty}\frac{\operatorname{ch}pa\,da}{\sqrt{(\operatorname{ch}^2\frac{1}{2}a-x)}}, \tag{7}$$

changing in (3), where $p=\sqrt{(N-\frac{1}{4})}$, to

$$F=A\int_0^{\pi}\frac{\cos pa\,da}{\sqrt{(\operatorname{ch}^2\frac{1}{2}a-x)}}. \tag{8}$$

Hobson's expression (*Phil. Trans.*, 1896) for the Mehler function of order p, solution of (4); and so a H. G. function of imaginary order, with

$$\alpha=\tfrac{1}{2}+pi,\quad \beta=\tfrac{1}{2}-pi,\quad \gamma=1,\quad x=\sin^2\tfrac{1}{2}\theta. \tag{9}$$

The Tesseral Harmonic (Hobson, *Phil. Trans.*, 1896),

$$P_{n,m}(\mu)=\frac{1}{\Pi(m)}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}}F\{-n,\ n+1,\ 1+m,\ \tfrac{1}{2}(1-\mu)\}\tag{10}$$

can then be expressed by the definite integral

$$\int_0^K\left(\frac{\operatorname{sn} v\operatorname{dn} v}{\operatorname{cn} v}\right)^{2n+1}(\operatorname{cn} v)^{2m}dv,\tag{11}$$

or

$$\int_0^{2K}\left(\frac{1-\operatorname{cn} w}{1+\operatorname{cn} w}\right)^{n+\frac{1}{2}}\left(\frac{\operatorname{dn} w+\operatorname{cn} w}{1+\operatorname{dn} w}\right)^m dw.\tag{12}$$

34. Comparing the expansion, in ascending powers of $\frac{r}{a}$ or $\frac{a}{r}$, of the typical terms $\frac{1}{R}$ and R of the P. F. and S. F., where

$$R^2=a^2-2ar\mu+r^2,\tag{1}$$

$$\frac{1}{R}=\Sigma\left(\frac{r^n}{a^{n+1}}\text{ or }\frac{a^n}{r^{n+1}}\right)P_n(\mu),\tag{2}$$

$$R=\Sigma\left(\frac{r^{n+1}}{a^n}\text{ or }\frac{a^{n+1}}{r^n}\right)I_n(\mu),\tag{3}$$

then $P_n(\mu)$ is the zonal harmonic, and $I_n(\mu)$ the associated function for the S. F., and then in (2), § 4, and (2), § 6, with

$$x=\tfrac{1}{2}(1-\mu)=\sin^2\tfrac{1}{2}\theta,\quad x'=\tfrac{1}{2}(1+\mu)=\cos^2\tfrac{1}{2}\theta,$$

$$\frac{d}{d\mu}(1-\mu^2)\frac{dP_n}{d\mu}+n(n+1)P_n=0,\quad \frac{d}{dx}\left(xx'\frac{dP_n}{dx}\right)+n(n+1)P_n=0,\tag{4}$$

$$(1-\mu^2)\frac{d^2I_n}{d\mu^2}+n(n+1)I_n=0,\quad xx'\frac{d^2I_n}{dx^2}+n(n+1)I_n=0,\tag{5}$$

so that

$$\frac{dI_n}{d\mu}=P_n,\quad I_n=\int P_n d\mu=-\frac{1-\mu^2}{n(n+1)}\frac{dP_n}{d\mu}=\frac{1}{n(n+1)}\sin\theta\frac{dP_n}{d\theta}.\tag{6}$$

And when n is an integer,

$$P_n=\frac{1}{n!}\frac{d^n(xx')^n}{dx'^n},\quad I_n=\frac{2}{n!}\left(\frac{d}{dx'}\right)^{n-1}(xx')^n,\tag{7}$$

$$(2n+1)I_n=\ P_{n+1}-P_{n-1},\tag{8}$$

$$I_n=-P_{n+1}+2\mu P_n-P_{n-1},\tag{9}$$

$$n(n+1)I_n=-(1-\mu^2)\frac{dP_n}{d\mu}=(n+1)(P_{n+1}-\mu P_n)=n(\mu P_n-P_{n-1}),\tag{10}$$

and so on, as discussed by Sampson on the "Stokes Current Function," *Phil. Trans.*, 1890, where the expressions given for I_n by the H. G. formulas can be replaced by definite integrals of the elliptic function.

The relations between a P. F. V and its S. F. N

$$r\sin\theta\frac{dV}{dr}=\frac{dN}{rd\theta},\quad r\sin\theta\frac{dV}{rd\theta}=-\frac{dN}{dr} \tag{11}$$

are satisfied then by typical terms, such as

$$(Ar^n+Br^{-n-1})P_n(\mu)\ \text{in}\ V,\ \text{and}\ [nAr^{n+1}+(n+1)Br^{-n}]I_n(\mu)\ \text{in}\ N; \tag{12}$$

and constant V and N represent orthogonal surfaces; and I_n may be replaced by $P_{n+1}-P_{n-1}$ in (8).

35. The Hicks toroidal function (*Phil. Trans.*, 1881–4) is of similar nature, defined by

$$P_m(u)=\int_0^\pi\frac{d\theta}{(\operatorname{ch}u+\operatorname{sh}u\cos\theta)^{m+\frac{1}{2}}}, \tag{1}$$

solution of the D. E.

$$\frac{d^2P}{du^2}+\coth u\frac{dP}{du}-(m^2-\tfrac{1}{4})P=0; \tag{2}$$

or writing C and S for ch u and sh u, and putting

$$C+1=2\operatorname{ch}^2\tfrac{1}{2}u=2x,\quad C-1=2\operatorname{sh}^2\tfrac{1}{2}u=2(x-1), \tag{3}$$

$$\frac{d^2P}{du^2}+\coth u\frac{dP}{du}=\frac{d^2P}{du^2}+\frac{C}{S}\frac{dP}{du}=\frac{1}{S}\frac{d}{du}\left(S\frac{dP}{du}\right)$$

$$=\frac{d}{dC}(C^2-1)\frac{dP}{dC}=\frac{d}{dx}x(x-1)\frac{dP}{dx}$$

$$=-x(1-x)\frac{d^2P}{dx^2}-(1-2x)\frac{dP}{dx}=(m^2-\tfrac{1}{4})P, \tag{4}$$

the H. G. D. E. with

$$\alpha+\beta=1,\quad \gamma=1,\quad \alpha\beta=-m^2+\tfrac{1}{4},\quad \alpha,\beta=\tfrac{1}{2}\pm m. \tag{5}$$

Then

$$C+S\cos\theta=C+S-2S\sin^2\tfrac{1}{2}\theta=e^u\Delta^2\tfrac{1}{2}\theta=\frac{\operatorname{dn}^2v}{\gamma'},$$

$$\tfrac{1}{2}\theta=\operatorname{am}v=\operatorname{am}eGe^{-u}=\gamma',\quad C=\tfrac{1}{2}\left(\frac{1}{\gamma'}+\gamma'\right),\quad S=\tfrac{1}{2}\left(\frac{1}{\gamma'}-\gamma'\right), \tag{6}$$

$$P_m(u)=\int_0^{\frac{1}{2}\pi}\frac{2d\tfrac{1}{2}\theta}{\left(\dfrac{\Delta\tfrac{1}{2}\theta}{\sqrt{\gamma'}}\right)^{2m+1}}=2\sqrt{\gamma'}\int_0^G\frac{dv}{\left(\dfrac{\operatorname{dn}v}{\sqrt{\gamma'}}\right)^{2m}}=2\sqrt{\gamma'}\int_0^G\left(\frac{\operatorname{dn}v}{\sqrt{\gamma'}}\right)^{2m}dv, \tag{7}$$

as the value is unaltered on changing v into $G-v$.

The sequence equation,

$$(2m+1)P_{m+1}-4mCP_m+(2m-1)P_{m-1}=0, \tag{8}$$

a difference equation with solution P_m, is then obtained by an integration between 0 and G of the differentiation

$$\frac{d}{dv}\frac{\gamma^2}{\gamma'}\operatorname{sn} v \operatorname{cn} v\left(\frac{\operatorname{dn} v}{\sqrt{\gamma'}}\right)^{2m-1}=(2m+1)\left(\frac{\operatorname{dn} v}{\sqrt{\gamma'}}\right)^{2m+2}$$
$$-4mC\left(\frac{\operatorname{dn} v}{\sqrt{\gamma'}}\right)^{2m}+(2m-1)\left(\frac{\operatorname{dn} v}{\sqrt{\gamma'}}\right)^{2m-2}. \tag{9}$$

Starting with

$$P_0=2\sqrt{\gamma'}F(\gamma),\ P_1=\frac{2E(\gamma)}{\sqrt{\gamma'}},\ \text{and then}\ P_2=\tfrac{2}{3}\left(\frac{1}{\gamma'}+\gamma'\right)\frac{2E(\gamma)}{\sqrt{\gamma'}}-\tfrac{2}{3}\sqrt{\gamma'}F(\gamma), \tag{10}$$

the sequence equation (8) will determine P_m for integral values of m.

36. In a quadric transformation of the toroidal function in (7), § 35, put

$$\varkappa=\operatorname{th}\tfrac{1}{2}u=\frac{1-\gamma'}{1+\gamma'},\quad C=\operatorname{ch} u=\frac{1+\varkappa^2}{1-\varkappa^2},\quad S=\frac{2\varkappa}{1-\varkappa^2}, \tag{1}$$

$$C+S\cos\theta=\frac{1+\varkappa\sin\psi}{1-\varkappa\sin\psi},\quad \cos\theta=\frac{\sin\psi-\varkappa}{1-\varkappa\sin\psi},$$
$$\sin\theta=\frac{\varkappa'\cos\psi}{1-\varkappa\sin\psi},\quad \tan^2\tfrac{1}{2}\theta=\frac{1+\varkappa}{1-\varkappa}\cdot\frac{1-\sin\psi}{1+\sin\psi}, \tag{2}$$

$$\frac{-d\theta}{\sqrt{(C+S\cos\theta)}}=\frac{\varkappa' d\psi}{\Delta(\psi,\varkappa)}=\varkappa' dw,\quad \psi=\operatorname{am}(w,\varkappa),\quad w=(1-2c)K, \tag{3}$$

$$P_m(u)=\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(\frac{1-\varkappa\sin\psi}{1+\varkappa\sin\psi}\right)^m\frac{\varkappa' d\psi}{\Delta\psi}=\int_{-L}^{L}\left(\frac{1-\varkappa\operatorname{sn} w}{1+\varkappa\operatorname{sn} w}\right)^m\varkappa' dw, \tag{4}$$

$$P_0=2\varkappa' F(\varkappa),\quad P_1=\frac{2E(\varkappa)-\varkappa'^2F(\varkappa)}{\varkappa'},\ \ldots., \tag{5}$$

$$\gamma'\operatorname{tn}^2 eG=\frac{1-\operatorname{sn}(1-2e)K}{1+\operatorname{sn}(1-2e)K},\quad \frac{\operatorname{dn}^2 eG}{\gamma'}=\frac{1+\varkappa\operatorname{sn}(1-2e)K}{1-\varkappa\operatorname{sn}(1-2e)K}. \tag{6}$$

This quadric transformation is illustrated geometrically on an ellipse of excentricity $\varkappa=\operatorname{th}\frac{1}{2}u$, in the connection between θ the focal angle or true anomaly from perihelion, and ψ the minor excentric angle or anomaly; and then with $v=eG$, $w=(1-2e)K$, $\theta=2\operatorname{am} eG$, $\psi=\operatorname{am}(1-2e)K$.

It is also shown on figure 5, in the expression of the potential of an anchor ring or torus, discussed by Dyson, *Phil. Trans.*, 1893.

37. In the reduction of the toroidal of the second kind

$$Q_m(u)=\int_0^\infty\frac{d\theta'}{(C+S\operatorname{ch}\theta')^{m+\frac{1}{2}}}, \tag{1}$$

substitute

$$\operatorname{ch}\tfrac{1}{2}\theta' = \sec\phi, \quad \operatorname{sh}\tfrac{1}{2}\theta' = \tan\phi, \quad d\theta' = 2\sec\phi d\phi, \tag{2}$$

$$C+S\operatorname{ch}\theta' = \frac{\Delta^2\phi}{\gamma'\cos^2\phi}, \text{ to modulus } \gamma' = e^{-u}, \tag{3}$$

$$\frac{d\theta'}{\sqrt{(C+S\operatorname{ch}\theta')}} = \frac{2\sqrt{\gamma'}d\phi}{\Delta\phi} = 2\sqrt{\gamma'}dv', \quad \phi = \operatorname{am}(v', \gamma') = \operatorname{am} fG', \tag{4}$$

$$\begin{aligned} Q_m(u) &= \int_0^{\frac{1}{2}\pi}\left(\frac{\sqrt{\gamma'}\cos\phi}{\Delta\phi}\right)^{2m}\frac{2\sqrt{\gamma'}d\phi}{\Delta\phi} \\ &= 2\gamma'^{m+\frac{1}{2}}\int_0^{G'}[\operatorname{sn}(G-v')]^{2m}dv' = 2\gamma'^{m+\frac{1}{2}}\int_0^{G'}(\operatorname{sn} v')^{2m}dv'. \end{aligned} \tag{5}$$

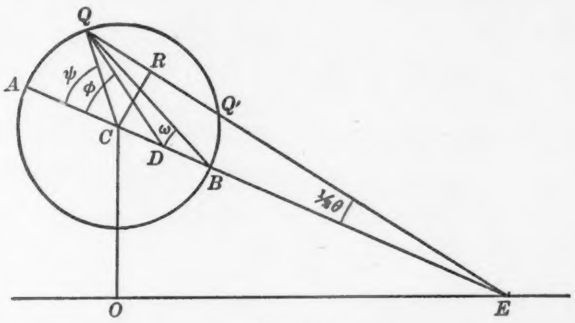

FIG. 5.

And with this toroidal in the form (this v not the same as in § 35, but employed with u, in the stereographic coordinates of A. J. M., § 24, p. 411), *i. e.* of the former volume.

$$Q_m(u) = \int_0^{\pi}\frac{\cos mv dv}{\sqrt{(2\cdot C+\cos v)}}, \tag{6}$$

$$\frac{dv}{\sqrt{(2\cdot C+\cos v)}} = \frac{\frac{1}{2}dv}{\sqrt{(\operatorname{ch}^2\frac{1}{2}u - \sin^2\frac{1}{2}v)}} = \frac{\varkappa' d\frac{1}{2}v}{\Delta(\frac{1}{2}v, \varkappa')} = \varkappa' dw',$$

$$\varkappa = \operatorname{sech}\tfrac{1}{2}u, \quad \varkappa = \operatorname{th}\tfrac{1}{2}u = \frac{1-\gamma'}{1+\gamma'}, \tag{7}$$

the complementary quadric transformation, with $\frac{1}{2}v = \operatorname{am}(w', \varkappa') = \operatorname{am} fK'$,

$$Q_m(u) = \int_0^{K'}\cos(2m\operatorname{am} w')dw'. \tag{8}$$

The connection between the reduction of (1) and (6) is made through

$$\cos\tfrac{1}{2}v = \frac{\sqrt{(C+S\operatorname{ch} v')}}{\operatorname{sh}\frac{1}{2}u + \operatorname{ch}\frac{1}{2}u\operatorname{ch} v'}, \quad \sin\tfrac{1}{2}v = \frac{\operatorname{ch}\frac{1}{2}u\operatorname{sh} v'}{\operatorname{sh}\frac{1}{2}u + \operatorname{ch}\frac{1}{2}u\operatorname{ch} v'}, \tag{9}$$

$$\frac{dv}{\sqrt{(2\cdot C+\cos v)}} = \frac{dv'}{\sqrt{(C+S\operatorname{ch} v')}}, \tag{10}$$

$$\operatorname{cn} fK' = \frac{\operatorname{cn} fG'\operatorname{dn} fG'}{1+\gamma'\operatorname{sn}^2 fG'}, \quad \operatorname{sn} fK' = \frac{(1+\gamma')\operatorname{sn} fG'}{1+\gamma'\operatorname{sn}^2 fG'}, \quad \operatorname{dn} fG' = \frac{1-\gamma'\operatorname{sn}^2 fG'}{1+\gamma'\operatorname{sn}^2 fG'}, \tag{11}$$

in the complementary quadric transformation to § 36.

Then Basset's function L_m ("Hydrodynamics" I, p. 107) defined, with $c=e^{-u}=\gamma'$, by

$$mL_m=\int_0^\pi \frac{\cos(m-1)v-\cos(m+1)v}{\sqrt{(2\cdot C+\cos v)}}\,dv = Q_{m-1}-Q_{m+1}, \tag{12}$$

$$(2m+3)L_{m+1}-4mCL_m+(2m-3)L_{m-1}=0. \tag{13}$$

A comparison can be made too with the expressions given in Todhunter's "Functions of Laplace, Lamé, and Bessel," Chapter IV.

38. For the potential V of a solid anchor ring of mass M at a point E on the axis (Dyson, *Phil. Trans.*, 1893; Routh, "Statics," II, p. 96), we take their expression:

$$\frac{V}{M}=\frac{2}{\pi}\int_0^\pi \frac{\sin^2\psi d\psi}{EQ}, \tag{1}$$

where on figure 5, $EC=A,\ CQ=a,\ OC=c,\ ACQ=\psi=2\omega$,

$$EQ^2=r^2=A^2+2aA\cos\psi+a^2=(A+a)^2\cos^2\omega+(A-a)^2\sin^2\omega, \tag{2}$$

and with $EA=A+a=r_1,\ EB=A-a=r_2$,

$$\frac{V}{M}=\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin^2 2\omega d\omega}{\sqrt{(r_1^2\cos^2\omega+r_2^2\sin^2\omega)}}. \tag{3}$$

With r for variable, and writing R for $r_1^2-r^2\cdot r^2-r_2^2$,

$$\sin^2\omega=\frac{r_1^2-r^2}{r_1^2-r_2^2},\quad \cos^2\omega=\frac{r^2-r_2^2}{r_1^2-r_2^2},\quad \frac{d\omega}{\sqrt{(r_1^2\cos^2\omega+r_2^2\sin^2\omega)}}=\frac{dr}{\sqrt{R}}, \tag{4}$$

$$\frac{V}{M}=\frac{16}{\pi(r_1^2-r_2^2)^2}\int_{r_2}^{r_1}\sqrt{R}dr. \tag{5}$$

Integrating by parts,

$$\begin{aligned}\int\sqrt{R}dr&=r\sqrt{R}-\int[(r_1^2+r_2^2)r^2-2r^4]\frac{dr}{\sqrt{R}}\\ &=r\sqrt{R}-2\int\sqrt{R}dr+(r_1^2+r_2^2)\int\frac{r^2dr}{\sqrt{R}}-2r_1^2r_2^2\int\frac{dr}{\sqrt{R}}\\ &=\tfrac{1}{3}r\sqrt{R}+\tfrac{1}{3}(r_1^2+r_2^2)\int\frac{r^2dr}{\sqrt{R}}-\tfrac{2}{3}r_1^2r_2^2\int\frac{dr}{\sqrt{R}}.\end{aligned} \tag{6}$$

Between the limits r_2 and r_1,

$$\sqrt{R}=0,\quad \int\frac{r^2dr}{\sqrt{R}}=r_1\int\Delta\omega d\omega=r_1E(\gamma),\quad \int\frac{dr}{\sqrt{R}}=\frac{F(\gamma)}{r_1},\quad \gamma'=\frac{r_2}{r_1}, \tag{7}$$

$$\frac{V}{M}=\frac{16}{3\pi r_1}\,\frac{(r_1^2+r_2^2)r_1^2E(\gamma)-2r_1^2r_2^2F(\gamma)}{(r_1^2-r_2^2)^2}=\frac{16}{3\pi r_1}\,\frac{(1+\gamma'^2)E(\gamma)-2\gamma'^2F(\gamma)}{\gamma^4}. \tag{8}$$

Or with a quadric transformation, as in Routh, "Statics," II, p. 96, with D, E inverse points in the circle AQB, and putting, as in pendulum motion,

$$CEQ=CQD=\tfrac{1}{2}\theta,\quad CDQ=CQE=\phi=\psi-\tfrac{1}{2}\theta,$$

$$\frac{V}{M}=\frac{2}{\pi A^2}\int\cos^2\phi\sqrt{(A^2-a^2\sin^2\phi)}\,d\phi=\frac{4}{\pi A}\int_0^{\frac{1}{2}\pi}\cos^2\phi\Delta(\phi,\varkappa)\,d\phi,$$

$$\varkappa=\frac{a}{A}=\frac{1-\gamma'}{1+\gamma'},\qquad(9)$$

$$\frac{V}{M}=\frac{4}{\pi A}\int_{e=0}^{1}\mathrm{cn}^2\,eK\,\mathrm{dn}^2\,eK\,deK=\frac{4}{\pi A}\int\left(\frac{\mathrm{dn}^4eK}{\varkappa^2}-\frac{\varkappa'^2\mathrm{dn}^2\,eK}{\varkappa^2}\right)deK$$

$$=\frac{4}{\pi A}\int_0^K\left[\frac{\varkappa'^2}{\varkappa^2}\left(\frac{\mathrm{dn}\,v}{\sqrt{\varkappa'}}\right)^4-\frac{\varkappa'^3}{\varkappa^2}\left(\frac{\mathrm{dn}\,v}{\sqrt{\varkappa'}}\right)^2\right]dv=\frac{4}{\pi A}\,\frac{1}{2\sqrt{\varkappa'}}\left(\frac{\varkappa'^2}{\varkappa^2}P_2-\frac{\varkappa'^3}{\varkappa^2}P_1\right),\qquad(10)$$

in the sequence equation (8), § 35, making

$$\frac{V}{M}=\frac{4}{3\pi A}\,\frac{2(1+\varkappa'^2)E(\varkappa)-\varkappa'^3F(\varkappa)}{\varkappa^2},\qquad(11)$$

a quadric transformation of the expression in (8).

So also for U, the skin potential of the anchor ring, if R is the midpoint of QQ',

$$\frac{U}{M}=\int\frac{PR}{\pi A^2}\,d\phi=\frac{1}{\pi A^2}\int_0^{\pi}\sqrt{(A^2-a^2\sin^2\phi)}\,d\phi=\frac{2}{\pi A}\int_0^{\frac{1}{2}\pi}\Delta\phi\,d\phi=\frac{2E(\varkappa)}{\pi A}.\qquad(12)$$

39. As shown on figure 1 and in § 37,

$$e^{-u}=\gamma'=\frac{PB}{PA}=\frac{EB}{EA}=\frac{1-\varkappa}{1+\varkappa},\quad \varkappa=\frac{OE}{OA}=\mathrm{th}\,\tfrac{1}{2}u,$$

$$\tfrac{1}{2}v=DEp=DPB=\mathrm{am}\,fK',\quad DEP=\mathrm{am}\,(1-f)K',$$

$$EBP=\mathrm{am}\,2fG',\quad DBP=\mathrm{am}\,2(1-f)G';$$

$$\Omega=2\pi(1-f)-4K\,\mathrm{zn}\,fK'=2\pi(1-f)-2G\,\mathrm{zn}\,2fG'-2G\gamma'\,\mathrm{sn}\,2fG'.$$

The point P may be supposed to circulate round the circle on DE in pendulum motion, with velocity due to the level of Ox, or proportional to BP or AP.

Starting from E, where $f=0$, $\Omega=2\pi$, as P moves along the semicircle ESD, f grows from 0 to 1, and Ω diminishes from 2π to 0. As P continues to complete the circuit back to E, f grows from 1 to 2, and Ω is negative, decreasing from 0 to -2π.

Thus 4π must be added to Ω in crossing the disc AB to make a fresh start; or P moving the other way round, clockwise, 4π must be subtracted in passing through E, as twelve hours is subtracted in passing through XII o'clock.

The angle $\phi=\mathrm{am}\ fG'$ in § 37 is constructed by bisecting the angle EBP in fig. 1 by By, crossing Dp in y, and then drawing the coaxial circle touching Dp

at y, on diameter de on DE, with limiting points A and B, and yd, ye bisect AyB externally and internally. Then, with centre at a, if dr, es are the tangents at d, e, the line rs passes through B, and crosses the circle on FB in q, at the same level as a, because Fq at right angles to Bq bisects rs.

Then with $DBp=\text{am}\,2fG'$, $DBr=\text{am}\,fG'=\phi$,

$$\gamma'=\frac{DB}{DA}=\frac{EB}{EA}=\frac{DB-EB}{DA-EA}=\frac{FB}{FE}, \quad Fa=FB\sin^2\phi=FE\cdot\gamma'\,\text{sn}^2 fG';$$

and with $DEp=\frac{1}{2}v=\text{am}\,fK'$,

$$\text{dn}\,fK'=\frac{Bp}{BD}=\frac{py}{yD}=\frac{Ea}{aD}=\frac{EF-Fa}{DF+Fa}=\frac{1-\gamma'\,\text{sn}^2 fG'}{1+\gamma'\,\text{sn}^2 fG}.$$

Again, by the property of coaxial circles,

$$\frac{Dy}{DB}=\frac{py}{pB}=\frac{rd}{rB}=\sin\phi; \quad Dy=DF(1+\gamma')\,\text{sn}\,fG',$$

$$\text{sn}\,fK'=\sin\tfrac{1}{2}v=\sin Day=\frac{Dy}{Da}=\frac{(1+\gamma')\,\text{sn}\,fG'}{1+\gamma'\,\text{sn}^2 fG'};$$

and

$$ay^2=aB\cdot aA=FB\,\text{cn}^2 fG'\cdot FA\,\text{dn}^2 fG'=FE^2\,\text{cn}^2 fG'\,\text{dn}^2 fG',$$

$$\text{cn}\,fK'=\cos DAy=\frac{ay}{Da}=\frac{\text{cn}\,fG'\,\text{dn}\,fG'}{1+\gamma'\,\text{sn}^2 fG'}.$$

In a transformation with the stereographic coordinates (u, v) of the point P, where

$$y+ix=a\,\text{th}\,\tfrac{1}{2}(u+iv), \quad y, x=\frac{a(\text{sh}\,u,\ \sin v)}{\text{ch}\,u+\cos v},$$

the circular arc APB, orthogonal to the circle DPE, will cross AB at the curvilinear angle $ABP=v=2\ \text{am}\,fK'=DFp$, while the rectilinear angle $ABP=\text{am}\,2fG'$.

40. Shown on figure 2 and in § 36,

$$x=\frac{OE}{OB}=\frac{1-\gamma'}{1+\gamma'}=\text{th}\,\tfrac{1}{2}u, \quad \tfrac{1}{2}\theta=\omega=ABQ=\text{am}\,eG, \quad AQE=ABq=\text{am}\,(1-e)G,$$

$$AQq'=\text{am}\,(1+e)G; \quad \text{and} \quad QE=AE\,\text{dn}\,eG, \quad Eq=AE\,\text{dn}\,(1-e)G.$$

The point Q may be supposed to circulate round the circle on AB in pendulum motion, with velocity due to the level of F, or proportional to EQ or DQ.

When turned about Oy into planes at right angles, the P and Q circles of figure 1 and 2 are linked as a magnetic and electric circuit.

The quadric transformation of § 36 is shown by drawing the perpendicular OR, AX on Qq; and then, with $OER=\chi=\operatorname{am} 2eK$,

$$OR=OE\sin\chi=OQ\cdot\varkappa\operatorname{sn}2eK,\quad QR=OQ\operatorname{dn}2eK,$$
$$ER=OE\cos\chi=OQ\cdot\varkappa\operatorname{cn}2eK,\quad RX=OQ\operatorname{cn}2eK;$$

and in (6) § 36,

$$\gamma'\operatorname{tn}^2 eG=\frac{\operatorname{tn} eG}{\operatorname{tn}(1-e)G}=\frac{\tan ABQ}{\tan ABq}$$

$$=\frac{\tan AqQ}{\tan AQq}=\frac{QX}{Xq}=\frac{QR-RX}{QR+RX}=\frac{\operatorname{dn}2eK-\operatorname{cn}2eK}{\operatorname{dn}2eK+\operatorname{cn}2eK},$$

$$\frac{\operatorname{dn}^2 eG}{\gamma'}=\frac{\operatorname{dn} eG}{\operatorname{dn}(1-e)G}=\frac{QE}{Eq}=\frac{QR+RE}{QR-RE}=\frac{\operatorname{dn}2eK+\varkappa\operatorname{cn}2eK}{\operatorname{dn}2eK-\varkappa\operatorname{cn}2eK}.$$

But if L is taken in OE such that $EL=\varkappa'\cdot LO$, then $LR=LO\operatorname{dn}2eK$; and if RL is produced to meet the circle on OE again in R',

$$OER'=ORR'=\operatorname{am}(1-2e)K;$$

and

$$LR\sin ORL=OL\cos\chi,\quad \sin ORR'=\frac{OL}{LR}\operatorname{cn}2eK=\frac{\operatorname{cn}2eK}{\operatorname{dn}2eK}=\operatorname{sn}(1-2e)K=\sin\psi;$$

and the angle ψ of the results in § 36 is shown in figure 2 by the angle ORL.

Continuing the quadric transformation,

$$OLR=\operatorname{am}4eL,\quad \lambda=\frac{1-\varkappa'}{1+\varkappa'}=\left(\frac{1-\sqrt{\gamma'}}{1+\sqrt{\gamma'}}\right)^2=\frac{CL}{CE},$$

if C is the centre of the circle on OE; and so on.

41. Maxwell obtains the expression of U, Ω, Ω' for a point Q on the axis, in a series of powers of $z=CQ$; and thence infers the series for a point P off the axis by introducing the zonal harmonic $Q_i(\phi)$ of the appropriate order i, as a factor of each term.

His method can be extended to the determination in the A. J. M. of W, W' (§ 3), $\frac{dV}{db}$ and V (§ 6), $\frac{dV}{d\gamma}$ (§ 15) and v (§ 16) for a thin lens; and an identification made of the result for a point Q on the axis, and thence generally for P off the axis.

The complicated dissection and integrations can then be avoided, required in G. W. Hill's method, although the results of his method must serve as a guide to a form, intangible and invisible otherwise.

Beginning with $\Omega(Q)$, the conical angle subtended at Q by the spherical segment on its circular base AB,

$$\frac{\Omega(Q)}{2\pi}=1-\frac{QO}{QA}=1-\frac{c\cos\gamma-z}{QA}=1-\left(\cos\gamma-\frac{z}{c}\right)\frac{c}{QA}$$

$$=1-\left(\cos\gamma-\frac{z}{c}\right)\left(1+\Sigma Q_i\frac{z^i}{c^i}\right), \tag{1}$$

writing Q_i for the zonal harmonic $Q_i(\gamma)$,

$$\frac{\Omega(Q)}{2\pi}=1-\cos\gamma+(Q_0-Q_1\cos\gamma)\frac{z}{c}+(Q_1-Q_2\cos\gamma)\frac{z^2}{c^2}$$

$$+\ldots.+(Q_{i-1}-Q_i\cos\gamma)\frac{z^i}{c^i}\ldots., \tag{2}$$

and with $z=0$, $\Omega(C)=2\pi(1-\cos\gamma)$.

But at Q', inverse point of Q, and with the other aspect of the spherical surface,

$$\frac{\Omega(Q')}{2\pi}=\frac{Q'O}{Q'A}-1=\frac{z'-c\cos\gamma}{Q'A}-1$$

$$=(z'-c\cos\gamma)\left(\frac{1}{z'}+Q_1\frac{c}{z'^2}+\ldots.+Q_i\frac{c^i}{z'^{i+1}}+\ldots.\right)-1$$

$$=(Q_2-Q_1\cos\gamma)\frac{c^2}{z'^2}+\ldots.+(Q_{i+1}-Q_i\cos\gamma)\frac{c^{i+1}}{z'^{i+1}}+\ldots., \tag{3}$$

and then in Ω' or $\Omega(P')$ at P', replace $\frac{c^{i+1}}{z'^{i+1}}$ by $\frac{c^{i+1}}{r'^{i+1}}Q_i(\phi)=\frac{r^{i+1}}{c^{i+1}}Q_i(\phi)$, so that

$$\frac{\Omega' r'}{2\pi c}=\Sigma(Q_{i+1}-Q_i\cos\gamma)\frac{r^i}{c^i}Q_i(\phi). \tag{4}$$

This makes

$$\frac{\Omega c+\Omega' r'}{2\pi}=c(1-\cos\gamma)+c\Sigma(Q_{i-1}-2Q_i\cos\gamma+Q_{i+1})\frac{r^i}{c^i}Q_i(\phi) \tag{5}$$

which agrees with Maxwell's result for $\frac{P}{2\pi}$ (E. and M., § 694), because

$$\int_\mu^1 Q_i d\mu=\frac{1-\mu^2}{i(i+1)}\frac{dQ_i}{d\mu}=\frac{\mu Q_i-Q_{i+1}}{i}=\frac{Q_{i-1}-Q_{i+1}}{2i+1}=Q_{i-1}-2\mu Q_i+Q_{i+1}, \tag{6}$$

and $\mu=\cos\gamma$. Also

$$\frac{P(Q)}{2\pi}=\frac{a}{QA}=\sin\gamma\left(1+\Sigma Q_i\frac{z^i}{c^i}\right),\quad \frac{P}{2\pi}=\sin\gamma\Sigma Q_i\frac{r^i}{c^i}Q_i(\phi). \tag{7}$$

Next for W, the P. F. of the circular plate AB (A. J. M., § 3),

$$\frac{W(Q)}{2\pi G\sigma}=\int_0^a\frac{ydy}{\sqrt{(y^2+QO^2)}}=QA-QO=\frac{QA^2}{QA}-QO$$

$$=c^2\left(1-2\frac{z}{c}\cos\gamma+\frac{z^2}{c^2}\right)\frac{1}{c}\left(Q_0+Q_1\frac{z}{c}+\ldots.+Q_i\frac{z^i}{c^i}\ldots.\right)$$

$$-c\cos\gamma+z, \tag{8}$$

$$\frac{W(Q)}{2\pi G\sigma c}=1-\cos\gamma+(1-\cos\gamma)\frac{z}{c}+(Q_0-2Q_1\cos\gamma+Q_2)\frac{z^2}{c^2}$$
$$+\ldots.+(Q_{i-2}-2Q_{i-1}\cos\gamma+Q_i)\frac{z^i}{c^i}\ldots., \tag{9}$$

$$\frac{W(Q')}{2\pi G\sigma}=Q'A-Q'O=\frac{Q'A^2}{Q'A}+c\cos\gamma-z'$$
$$=z'^2\left(1-2\frac{c}{z'}\cos\gamma+\frac{c^2}{z'^2}\right)\frac{1}{z'}\left(Q_0+Q_1\frac{c}{z'}+\ldots.+Q_i\frac{c^i}{z'^i}\ldots.\right)$$
$$+c\cos\gamma-z'=(Q_0-2Q_1\cos\gamma+Q_2)\frac{c^2}{z'}$$
$$+\ldots.+(Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{c^{i+2}}{z'^{i+1}}\ldots., \tag{10}$$

$$\frac{W(Q')z'}{2\pi G\sigma c^2}=\tfrac{1}{2}\sin^2\gamma+(Q_1-2Q_2\cos\gamma+Q_3)\frac{c}{z'}\ldots.$$
$$+(Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{c^i}{z'^i}\ldots. \tag{11}$$

$$\frac{W(Q)c+W(Q')z'}{2\pi G\sigma c^2}=\tfrac{1}{2}(1-\cos\gamma)(3+\cos\gamma)+\tfrac{1}{2}(1-\cos\gamma)^2(2+\cos\gamma)\frac{z}{c}\ldots.$$
$$+(Q_{i-2}-2Q_{i-1}\cos\gamma+2Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{z^i}{c^i}\ldots., \tag{12}$$

$$\frac{Wc+W'r'}{2\pi G\sigma c^2}=\Sigma(Q_{i-2}-2Q_{i-1}\cos\gamma+2Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{r^i}{c^i}Q_i(\phi). \tag{13}$$

So far all these functions are P. F.'s, P, Ω, Ω', U, W, W'; but going back to A. J. M., § 3, for the expression of W in terms of Pa, QA, Ωb, here QA is a S. F., given we find by

$$QA=Pa-\Omega b-\frac{W}{G\sigma}=c\Sigma(Q_{i-2}-Q_{i-1}\cos\gamma)(P_{i-1}\cos\phi-P_i)\frac{r^i}{c^i}$$
$$=c\Sigma\frac{\sin^2\gamma\sin^2\phi Q'_{i-1}P'_{i-1}(\phi)}{(i-1)i}\frac{r^i}{c^i}. \tag{14}$$

42. The expression in (4), § 6, A. J. M., p. 384, of the axial component of the attraction of the plano-convex lens is the equivalent of the potential of the base AB, coated with density σ, less the potential of the spherical surface, coated at E with density $\sigma\mu$, $\mu=\cos OCE$. It will serve too for the magnetic potential of the lens, or equivalent current sheet round the portion of the spherical surface.

The thin lens of § 15 may then be considered a spherical segment, coated with density $\sigma(\mu-\cos\gamma)$.

The result of (4), § 6, shows that the coating $\sigma\mu$ would have potential $U(\sigma\mu)$, which can be written

$$\frac{U(\sigma\mu)}{G\sigma}=\frac{W}{G\sigma}-\frac{1}{G\rho}\frac{dV}{db}=\frac{Wc+W'r'}{3G\sigma c}+\frac{U(\sigma)\cos\gamma}{3G\sigma}$$
$$=\tfrac{1}{3}\left(\frac{W}{G\sigma}+\Omega c\cos\gamma\right)+\tfrac{1}{3}\left(\frac{W'}{G\sigma}+\Omega'c\cos\gamma\right)\frac{r'}{c}. \tag{1}$$

There is an opportunity for a geometrical interpretation on the lines of that given for $U(\sigma)$ in § 3, but a difficulty is to account for the factor $\frac{1}{3}$.

To verify this in Maxwell's manner at a point Q on the axis, we have to evaluate

$$\frac{U(\sigma\mu, Q)}{2\pi G\sigma}=\int_{\cos\gamma}^{1}\frac{\mu d\mu}{\sqrt{(c^2-2cz\cos\gamma+z^2)}}=\frac{1}{c}\int\left(1+\Sigma Q_i\frac{z^i}{c^i}\right)\mu d\mu;$$

requiring $\int_\mu^1 Q_i\mu d\mu.$ (2)

Mr. J. R. Wilton gives me the general formula

$$\int_\mu^1 Q_iQ_jd\mu=\frac{(1-\mu^2)(Q_i'Q_j-Q_iQ_j')}{(i-j)(i+j+1)}. \tag{3}$$

But if $i=j$, we must make use of Hargreave's recurring formula (Whittaker, *Analysis*, p. 212).

$$(2i+1)Q_i^2-(2i-1)Q_{i-1}^2=\frac{d}{d\mu}[(Q_i^2+Q_{i-1}^2)-2Q_iQ_{i-1}]. \tag{4}$$

We only require here the special case of $j=1$, and then make use of the ordinary formulas,

$$iQ_{i-1}-(2i+1)Q_i\mu+(i+1)Q_{i+1}=0, \tag{5}$$

$$(2i+1)\int_\mu^1 Q_i\ d\mu=Q_{i-1}-Q_{i+1}=\frac{(1-\mu^2)Q_i'}{i(i+1)}, \tag{6}$$

$$(2i+1)\int_\mu^1 Q_i\mu d\mu=i\int Q_{i-1}d\mu+(i+1)\int Q_{i+1}d\mu; \tag{7}$$

and thence we find, after reduction,

$$\int_{\cos\gamma}^1 Q_i\mu d\mu=\tfrac{1}{3}(Q_{i-2}-Q_{i-1}\cos\gamma+2Q_i\sin^2\gamma-Q_{i+1}\cos\gamma+Q_{i+2}),$$

with $\int Q_0\mu d\mu=\frac{1}{2}\sin^2\gamma,\ \int Q_1\mu d\mu=\frac{1}{3}(1-\cos^3\gamma).$ (8)

But according to the preceding expressions in (5) and (13)

$$\frac{W(Q)c+W(Q')z'}{2\pi G\sigma c}+\frac{U(Q)\cos\gamma}{G\sigma}$$
$$=\tfrac{1}{2}c(1-\cos\gamma)(3+\cos\gamma)+\tfrac{1}{2}(1-\cos\gamma)(2+\cos\gamma+\cos^2\gamma)z\ldots.$$
$$+(Q_{i-2}-2Q_{i-1}\cos\gamma+2Q_i-2Q_{i+1}\cos\gamma+Q_{i+2})\frac{z^i}{c^{i-1}}\ldots.$$
$$+c\cos\gamma(1-\cos\gamma)+\tfrac{1}{2}z\cos\gamma\sin^2\gamma\ldots.$$
$$+\cos\gamma(Q_{i-1}-2Q\cos\gamma+Q_{i+1})\frac{z^i}{c^{i-1}}\ldots.$$
$$=\tfrac{3}{2}c\sin^2\gamma+(1-\cos^3\gamma)z\ldots.$$
$$+(Q_{i-2}-Q_{i-1}\cos\gamma+2Q_i\sin^2\gamma-Q_{i+1}\cos\gamma+Q_{i+2})\frac{z^i}{c^{i-1}}\ldots, \tag{9}$$

which adds the audit up, and so the identification is complete.

Then for v, the P. F. of the thin curved lens of § 15, treated as a spherical segment, coated with density $\sigma(\mu-\cos\gamma)$,

$$\frac{v}{G\sigma}=\frac{Wc+W'r'-2Uc\cos\gamma}{3G\sigma c},\quad \frac{v}{Gm}=\frac{Wc+W'r'-2Uc\cos\gamma}{3\pi G\sigma c^3(1-\cos\gamma)^2}, \tag{10}$$

if of mass $m=2\pi\sigma c^2\int_{\cos\gamma}^1(\mu-\cos\gamma)\,d\mu=\pi\sigma c^2(1-\cos\gamma)^2$.

For a flat lens, with $\gamma=0$, this expression for v takes an indeterminate form, and is best evaluated independently, as in § 16, by a dissection of concentric circles and radiating straight lines.

The P. F. W of the flat disc may be evaluated at the same time in this manner, and then

$$\frac{1}{G\sigma}\frac{dW}{d\theta}=\int_0^a\frac{y\,d\theta}{PQ'}=PQ-PO-IA\cos\theta, \tag{11}$$

$$I=\int_0^a\frac{dy}{PQ'}=\mathrm{ch}^{-1}\frac{PQ}{PZ}-\mathrm{ch}^{-1}\frac{PO}{PZ}=\mathrm{sh}^{-1}\frac{c+A\cos\theta}{PZ}-\mathrm{sh}^{-1}\frac{A\cos\theta}{PZ}, \tag{12}$$

$$PQ'^2=y^2+2Ay\cos\theta+A^2+b^2,\quad PQ^2=a^2+2Aa\cos\theta+A^2+b^2,$$
$$PZ^2=A^2\sin^2\theta+b^2,\quad PO^2=A^2+b^2, \tag{13}$$

$$\frac{dI}{d\theta}=-\frac{Aa\cos\theta+A^2+b^2}{PZ^2}\cdot\frac{A\sin\theta}{PQ}+\frac{PO\cdot A\sin\theta}{PZ^2} \tag{14}$$

43. The integral $\int I\,d\theta$ is intractable; it arises in the skin potential of the curved wall of a circular cylinder, and is shown graphically by a quadrature on the Mercator chart, as explained in the *Trans. American Math. Society*, October, 1907, §§ 53, 66.

But $\int IA\cos\theta\,d\theta$ is tractable, and integrating by parts

$$\int_0^{2\pi}IA\cos\theta\,d\theta=(IA\sin)_0^{2\pi}-\int\frac{dI}{d\theta}A\sin\theta\,d\theta$$
$$=\int\frac{Aa\cos\theta+A^2+b^2}{PZ^2}\cdot\frac{b^2d\theta}{PQ}-\int(Aa\cos\theta+A^2+b^2)\frac{d\theta}{PQ}$$
$$+PO\int\left(1-\frac{b^2}{PZ^2}\right)d\theta, \tag{15}$$

in which the first integral is $(2\pi-\Omega)b$, as shown in the *Trans. American Math. Society*, § 48, by a dissection of the circular area AB into sector elements $\frac{1}{2}a^2d\theta$, and then

$$\frac{W}{G\sigma}=\int PQ\,d\theta-2\pi PO+(2\pi-\Omega)b+QA-\frac{A^2+b^2}{a}\int\frac{d\theta}{PQ}+2\pi\cdot PO-2\pi b$$
$$=\frac{A^2+a^2+b^2}{a}P-2QA+(2\pi-\Omega)b+QA-\frac{A^2+b^2}{a}P-2\pi b$$
$$=Pa-QA-\Omega b, \tag{16}$$

as before.

The flat lens may be considered a disc, coated with density

$$\sigma\left(1-\frac{y^2}{a^2}\right)=\sigma\frac{AQ'\cdot Q'B}{OA^2},$$

and its potential is shown in § 16 to be expressible by the three elliptic integrals, of the I, II, and III kind, P, Q, and Ω.

A formula of integration by parts will show that a similar expression is obtained for a density $\sigma\left(1-\frac{y^2}{a^2}\right)^n$, or otherwise for a density varying as some even power of y.

But for an odd power, the intractable integral $\int I d\theta$ arises, in addition to the elliptic integrals.

A formula of reduction is obtained by the integration of the relation

$$\frac{d}{dy}(y^n PQ')=[(n+1)y^{n+1}+(2n+1)y^n A\cos\theta+ny^{n-1}(A^2+b^2)]\frac{1}{PQ'}. \quad (17)$$

In the definite integral the algebraical part vanishes at both limits.

Mr. Bromwich has expressed the results in a series in the *Proc. London Math. Society*, September, 1912.

On the electrified disc, and on a flattened oblate spheroid, the density is $\sigma\left(1-\frac{y^2}{a^2}\right)^n$, with $n=-\frac{1}{2}, \frac{1}{2}$; and the result for the potential is non-elliptic and given already. A similar formula of reduction will give the result for a density $\sigma\left(1-\frac{y^2}{a^2}\right)^{i-\frac{1}{2}}$, where i is an integer; and the evaluation should attract a careful worker, when the need arises in a physical problem.

So also for the S. F. of these coatings of superficial density, to be worked out as an exercise.

STAPLE INN, LONDON, *October* 12, 1915.

*On the Asymptotic Character of Functions Defined by Series of the Form $\Sigma c_n g(x+n)$.**

BY R. D. CARMICHAEL.

Introduction.

In a previous memoir † I have laid the foundations of a general theory of series of the forms

$$\Omega(x) = \sum_{n=0}^{\infty} c_n g(x+n), \tag{1}$$

$$\overline{\Omega}(x) = \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}, \tag{1'}$$

where $c_0, c_1, c_2, \ldots$ are constants and $g(x)$ is a function of x having the asymptotic character

$$g(x) \sim h(x)\left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \ldots\right), \qquad h(x) = x^{P(x)} e^{Q(x)}, \tag{2}$$

$P(x)$ and $Q(x)$ being polynomials which we write in the form

$$P(x) = \mu_0 + \mu_1 x + \mu_2 x^2 + \ldots + \mu_k x^k, \qquad \mu_k \neq 0 \text{ if } k > 0;$$
$$Q(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_m x^m, \qquad \alpha_m \neq 0 \text{ if } m > 0.$$

(In case $k=0$ we assume that $m>1$.) In I we required that the asymptotic relation (2) should be valid only for x approaching infinity in a positive sense along any line whatever parallel to the axis of reals. But for the application of series $\Omega(x)$ and $\overline{\Omega}(x)$ to a study of the properties of functions in the neighborhood of a singular point, it is convenient to place a stronger condition on $g(x)$ and to require, in fact, that the asymptotic relation (2) shall be valid when x is confined in any way to a sector V including the positive axis of reals in its interior. Furthermore, we shall suppose that the singularities of $g(x)$ in the finite plane are isolated points, and that $g(x)$ is analytic in V at every point for which $|x|$ is not less than a given constant K.

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* Presented to the American Mathematical Society (at Chicago), April 21, 1916.

† *Transactions of the American Mathematical Society*, Vol. XVII (1916), pp. 207–232. This paper will be referred to as *I*. For the most part we employ here the notation of the earlier memoir.

We employ σ to denote α_m or μ_k according as m is or is not greater than k and write

$$\sigma=\sigma_0+\sigma_1\sqrt{-1},$$

where σ_0 and σ_1 are real. If $\lambda[\mu]$ is a constant determined as in Theorem XII of *I*, then the region of convergence [absolute convergence] of $\Omega(x)$ is bounded by the straight line $R(\sigma x)=\lambda[R(\sigma x)=\mu]$ and lies on that side of this line for which $R(\sigma x)<\lambda[R(\sigma x)<\mu]$. According to Theorem XVI of *I*, a given function $f(x)$ can not have more than one expansion in a series $\Omega(x)$ for a given $g(x)$, provided that $R(\sigma)$ is negative; but such an expansion is not necessarily unique when the latter condition is not satisfied. Accordingly, we shall now further restrict $g(x)$ by requiring that $R(\sigma)$ shall be negative. We shall also suppose that non-exceptional points for both $\Omega(x)$ and $\overline{\Omega}(x)$ exist in every strip parallel to the line $R(\sigma x)=0$.

The general object of this paper is to determine the asymptotic character of a function $\Omega(x)$ defined by a series of the form (1). In § 1 a somewhat extended discussion results in the fundamental Theorem I by which the asymptotic character of the function $\Omega(x)$ in relation to $g(x)$ is determined. This result is quite satisfying from the point of view of its elegance and simplicity. For the important case when $k=1$ and $m=0$ or 1, Theorem I may be stated in the somewhat more convenient form of Theorem II. The character of the convergence of the series $\overline{\Omega}(x)$ with respect to uniformity is treated in § 2, the result being stated as Theorem III.

Guided by the result contained in Theorem I, I introduce in § 3 an extension of the notion of asymptotic representation. In § 4 it is shown that a special case of this extended notion includes as a special case the notion of asymptotic representation in the sense of Poincaré. In § 5 a further treatment is given of this new asymptotic representation in the case when $k=1$, $m=0$ or 1 and $\sigma=-1$. It is shown in this case that the notion is equivalent to that of Poincaré, and explicit formulae are obtained exhibiting this equivalence. Finally, in § 6, an illustrative example is given to emphasize the fact that an $\overline{\Omega}$-series is capable of representing conveniently the asymptotic character of certain functions near infinity, while at the same time it yields also the essential properties of these functions in the finite plane.

§ 1. *Asymptotic Character of the Function $\Omega(x)$ in Relation to $g(x)$.*

At the close of *I* we saw incidentally that the function $\Omega(x)$ defined by the series in equation (1) has the properties expressed in the relations

$$\lim\{g(x+s)\}^{-1}\{\Omega(x)-\sum_{n=0}^{s} c_n g(x+n)\}=0, \qquad s=0,1,2,\ldots, \tag{3}$$

provided that the limits are taken for x approaching infinity in a positive sense along the axis of reals. We now make inquiry as to what less stringent restrictions may be placed on the approach of x to infinity without destroying the general validity of relations (3). Naturally we make the general restriction that x shall remain in V as it approaches infinity.

In the first place, if relations (3) are to be valid in the special case when c_1 is different from zero and all the coefficients $c_2, c_3, \ldots.$ in (1) are equal to zero, it is necessary that the ratio $g(x+1)/g(x)$ shall approach zero as x approaches infinity in the manner to be specified. Moreover, if the region over which x may vary in approaching infinity is such that $x+n$ is in the region when x is in the region, n being a positive integer, then it is easy to see that the foregoing condition is also sufficient in all cases in which the series in (1) has only a finite number of coefficients different from zero, since

$$\frac{g(x+n)}{g(x+s)}=\frac{g(x+s+1)}{g(x+s)}\cdot\frac{g(x+s+2)}{g(x+s+1)}\cdot\ldots\cdot\frac{g(x+n)}{g(x+n-1)} \tag{4}$$

when n is any integer greater than the integer s. Accordingly, we are led to ascertain first the conditions under which $g(x+1)/g(x)$ shall approach zero as x approaches infinity.

We may write

$$\frac{g(x+1)}{g(x)}\sim e^{[P(x+1)-P(x)]\log x}e^{Q(x+1)-Q(x)}e^{P(x+1)\log\left(1+\frac{1}{x}\right)}\left(1-\frac{a_1}{x^2}+\ldots.\right),$$

the relation being valid for x approaching infinity in V. From this it follows readily that $g(x+1)/g(x)$ will approach zero when and only when

$$e^{P_1(x)\log x+Q_1(x)}$$

approaches zero, x being confined to V and $P_1(x)$ and $Q_1(x)$ being polynomials with the values

$$P_1(x)=P(x+1)-P(x),$$
$$Q_1(x)=Q(x+1)-Q(x)+\frac{1}{x}p_1(x)-\frac{1}{2x^2}p_2(x)$$
$$+\frac{1}{3x^3}p_3(x)-\ldots.+(-1)^{k-1}\frac{1}{kx^k}p_k(x),$$

where $p_j(x)$ denotes the sum of the terms of degree j or greater in the function $P(x+1)$ when arranged as a polynomial in x. Moreover, if we write

$$\frac{g(x+1)}{g(x)}=e^{P_1(x)\log x+Q_1(x)}\phi(x), \tag{5}$$

then $\phi(x)$ approaches unity as x approaches infinity in V.

The leading term in the polynomial $P_1(x)$ obviously is of degree $k-1$ and has the coefficient $k\mu_k$. If $m>k$, the leading term in the polynomial $Q_1(x)$ is

of degree $m-1$ and has the coefficient $m\alpha_m$. Now, if we put $x=\rho e^{i\theta}$ where ρ is positive and write

$$R\{P_1(x)\log x+Q_1(x)\}=\pi_1(\rho)\log\rho+\pi_2(\rho),$$

we see that $\pi_1(\rho)$ and $\pi_2(\rho)$ are polynomials in ρ with coefficients involving θ. If $k\geqq m$, so that μ_k is denoted by σ, then the coefficient of the highest power of ρ in $\pi_1(\rho)$ is $k(\sigma_0\cos t\theta-\sigma_1\sin t\theta)$, where $t=k-1$. If $k<m$, so that α_m is denoted by σ, then the coefficient of the highest power of ρ in $\pi_2(\rho)$ is $m(\sigma_0\cos t\theta-\sigma_1\sin t\theta)$, where $t=m-1$. If θ is held fixed and ρ is allowed to approach $+\infty$, then the absolute value of the exponential function in (5) approaches zero or infinity according as $\sigma_0\cos t\theta-\sigma_1\sin t\theta$ is less than zero or greater than zero; and when the last quantity is equal to zero, the absolute value of the exponential quantity in (5) may approach infinity or may approach zero owing to different possible forms of $P_1(x)$ and $Q_1(x)$.

From these considerations we are led to restrict x to the largest sector V_1 which includes the positive axis of reals, is in V and is such that

$$\sigma_0\cos t\theta-\sigma_1\sin t\theta\leqq-\varepsilon \tag{6}$$

for every x in V_1, ε being a positive quantity as small as one pleases, and t being equal to $m-1$ or $k-1$ according as m is or is not greater than k. It is clear that $g(x+1)/g(x)$ approaches zero as x approaches infinity in V_1. (In case $k=1$ and $m=0$ or 1, so that $t=0$, it is obvious that ε may be chosen so that (6) is satisfied for all values of θ; in this case V_1 coincides with V.)

Returning now to a consideration of the series $\Omega(x)$, let us denote its convergence number by λ so that the boundary of its region of convergence is the straight line $R(\sigma x)=\lambda$. Let λ_1 be any number less than λ, and confine x so that $R(\sigma x)\leqq\lambda_1$. Denote by $\overline{V}$ the maximum region which is common to V_1 and the half-plane $R(\sigma x)\leqq\lambda_1$. Obviously $\overline{V}$ contains the positive axis of reals in its interior, except possibly for a segment of finite length adjacent to the point 0.

Let x_0 be a non-exceptional point for the series $\Omega(x)$ subject to the condition that

$$\lambda_1<R(\sigma x_0)<\lambda.$$

Put

$$u_n=c_ng(x_0+n),\quad v_n=\frac{g(x+n)}{g(x_0+n)},\quad n\geqq l,$$

the integer l being chosen so that $g(x_0+n)$ is different from zero for every n not less than l. Then, if we write

$$U_{l-1}=0;\quad U_n=u_l+u_{l+1}+\ldots.+u_n,\quad n\geqq l,$$

we have

$$\begin{aligned}\sum_{n=l}^{m} c_n g(x+n) &= \sum_{n=l}^{m} u_n v_n \\ &= \sum_{n=l}^{m}(U_n-U_{n-1})v_n \\ &= -\sum_{n=l}^{m} U_n(v_{n+1}-v_n)+U_m v_{m+1}.\end{aligned}$$

From the convergence of $\Omega(x_0)$ it follows that U_n approaches a finite limit as n increases indefinitely. If x is held fixed while m increases indefinitely, it is clear that V_m approaches zero. (Compare (10) of I and the discussion immediately following.) Hence it follows from the last equation that a constant M exists (independent of x) such that for every non-exceptional x in the half-plane $R(\sigma x)\leqq\lambda_1$ we have

$$\left|\sum_{n=l}^{\infty} c_n g(x+n)\right|\leqq M\sum_{n=l}^{\infty}|v_{n+1}-v_n|, \tag{7}$$

the latter series being certainly convergent as we saw in I.

In view of (1) relations (3) may be expressed in the form

$$\lim \sum_{n=s+1}^{\infty} c_n\frac{g(x+n)}{g(x+s)}=0,\qquad s=0,1,2,\ldots.. \tag{8}$$

Let s_1 be a fixed integer and let s be any integer less than s_1. Then we may write

$$\sum_{n=s+1}^{\infty} c_n\frac{g(x+n)}{g(x+s)}=\sum_{n=s+1}^{s_1} c_n\frac{g(x+n)}{g(x+s)}+\frac{g(x+s_1)}{g(x+s)}\sum_{n=s_1+1}^{\infty} c_n\frac{g(x+n)}{g(x+s_1)}.$$

Now, suppose that x approaches infinity in $\overline{V}$ in such wise that (8) is valid for the particular value s_1 of s. Then from the last relation it follows readily that (8) is also valid, for such approach of x to infinity, whenever s is less than s_1. Therefore, to prove all the relations (8), and hence all the relations (3), valid for a given approach of x to infinity in $\overline{V}$, it is sufficient to prove them valid for those values of s which are greater than some given number.

Employing the result of the preceding paragraph and comparing (7) and (8) we see that all the relations (3) will be satisfied provided that x approaches infinity in $\overline{V}$ in such a way that

$$\lim\frac{1}{g(x+s)}\sum_{n=s+1}^{\infty}|v_{n+1}-v_n|=0 \tag{9}$$

for all values of the integer s greater than a conveniently chosen positive integer s_1. We choose s_1 so that it is not less than l. It is obvious that (9) may be replaced by the following equivalent relations:

$$\lim\sum_{n=s+1}^{\infty}\left|\frac{g(x_0+s)}{g(x+s)}\cdot\frac{g(x+n)}{g(x_0+n)}\left\{\frac{g(x+n+1)}{g(x_0+n+1)}\cdot\frac{g(x_0+n)}{g(x+n)}-1\right\}\right|=0,\quad s>s_1. \tag{10}$$

We are now in position to prove the following theorem:

THEOREM I. *Let $\Omega(x)$ be a function defined by a series (1) whose convergence number λ is not $-\infty$. Let t denote $m-1$ or $k-1$ according as m is or is not greater than k. Let $\bar{V}$ denote the greatest region of the x-plane common to V, the half-plane $R(\sigma x)\leqq\lambda_1$ and that sector $\sigma_0\cos t\theta-\sigma_1\sin t\theta\leqq-\varepsilon$ which includes the positive axis of reals, where λ_1 is any number less than λ, and ε is any positive number however small and θ is defined by the relation $x=\rho e^{i\theta}$, ρ being positive. Then if x approaches infinity in $\bar{V}$ we have*

$$\lim\{g(x+s)\}^{-1}\{\Omega(x)-\sum_{n=0}^{s}c_n g(x+n)\}=0,\qquad s=0,1,2,\ldots\ldots$$

If in (10) we replace g in part by h it is easy to see from the foregoing results that the proof of this theorem will be complete if we show that

$$\lim\frac{h(x_0+s)}{h(x_0+s+1)}\cdot\frac{h(x+s+1)}{h(x+s)}\sum_{n=s+1}^{\infty}\left|\frac{h(x_0+s+1)h(x+n)}{h(x+s+1)h(x_0+n)}\right|\cdot\left|\frac{g(x+n+1)g(x_0+n)}{g(x+n)g(x_0+n+1)}-1\right|=0,\qquad s>s_1,\quad(11)$$

the limit being taken for x approaching infinity in $\bar{V}$; and hence if we show that the sum of the series in (11) is bounded.

For this series in (11) we use for the moment the symbol $T(x,s)$. By $\bar{T}(x,s)$ we denote the series

$$\bar{T}(x,s)=\sum_{n=s+1}^{\infty}\left|\frac{h(x_0+s+1)h(x+n)}{h(x+s+1)h(x_0+n)}\right|\cdot\left\{\left|\frac{g(x+n+1)g(x_0+n)}{g(x+n)g(x_0+n+1)}\right|+1\right\}.$$

Then the series $T(x,s)$ is term by term less than or equal to the series $\bar{T}(x,s)$.

In case k or m is greater than unity it will now be shown that $\bar{T}(x,s)$ is bounded as x approaches infinity in $\bar{V}$; and thus the proof of the theorem for this case will be completed.

Let x_1 be a non-exceptional point for the series $\Omega(x)$ such that $\lambda_1<R(\sigma x_1)<R(\sigma x_0)$. Then the series $\bar{T}(x,s)$ converges whatever the value of s (greater than s_1). Therefore, in order to complete the proof of the theorem in this case it is sufficient (compare equation (4)) to show that a number X exists such that

$$\frac{h(x+j+1)}{h(x_1+j+1)}\cdot\frac{h(x_1+j)}{h(x+j)}\qquad(12)$$

is in absolute value less than 1 for every integer j, provided that $|x|>X$ and x is in $\bar{V}$. This we shall now prove without the restriction that k or m is greater than unity.

If we denote the expression in (12) by E and employ the notation $P_1(x)$ and $Q_1(x)$ introduced in the earlier part of this section, then we may write:

$$E=\frac{(x+j)^{P_1(x+j)}\left(1+\dfrac{1}{x+j}\right)^{P(x+j+1)}}{(x_1+j)^{P_1(x_1+j)}\left(1+\dfrac{1}{x_1+j}\right)^{P(x_1+j+1)}}\,e^{Q(x+j+1)-Q(x_1+j+1)-Q(x+j)+Q(x_1+j)};$$

or

$$E=(x+j)^{P_1(x+j)}(x_1+j)^{-P_1(x_1+j)}e^{Q_1(x+j)-Q_1(x_1+j)}S\equiv\overline{E}S,$$

where S is a quantity which approaches unity if either x or j approaches infinity, or if both x and j approach infinity in an independent manner, x remaining in $\overline{V}$. We may write

$$\overline{E}=\frac{|x+j|^{P_1(x+j)}\,e^{iP_1(x+j)\arg(x+j)}}{|x_1+j|^{P_1(x_1+j)}e^{iP_1(x_1+j)\arg(x_1+j)}}\,e^{Q_1(x+j)-Q_1(x_1+j)}.$$

We may write $P_1(x+j)$ as a polynomial in j in the form *

$$P_1(x+j)=\pi_0(x)+\pi_1(x)j+\ldots.+\pi_{k-1}(x)j^{k-1}.$$

We introduce the notation

$$\overline{P}_1(x+j)=|\pi_0(x)|+|\pi_1(x)|j+\ldots.+|\pi_{k-1}(x)|j^{k-1}.$$

Now, since x must remain in $\overline{V}$ it is easy to see that $|x+j|/|x_1+j|$ is bounded away from zero,† however x and j vary, provided that $|x|$ is sufficiently large. Hence a constant M exists such that

$$|\overline{E}|\leqq|x+j|^{R[P_1(x+j)]+\overline{P}_1(x_1+j)}e^{M[\overline{P}_1(x+j)+\overline{P}_1(x_1+j)]}e^{R[Q_1(x+j)-Q_1(x_1+j)]}\tag{13}$$

provided that $|x|$ is sufficiently large.

It is convenient now to separate into cases owing to the relative magnitude of k and m.

First, let us suppose that $k\geqq m$. Then (as we have seen) the leading term of $P_1(x)$, considered as a polynomial in x, is $k\sigma x^{k-1}$. With this in hand, let us consider $P_1(x+j)$ as a polynomial in j whose coefficients are arranged

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* The treatment in the text applies strictly only when $k>0$; but with the understanding that $P_1(x)\equiv 0$ and $\overline{P}_1(x)\equiv 0$ when $k=0$ the formulae employed are valid.

† This is obvious in case σ_1 is zero or j is bounded. When σ_1 is not zero and j is not bounded it may be proved as follows: Take j greater than $|x_1|$. Then the ratio in consideration is greater than $\frac{1}{4}$ if $u\geqq-\frac{1}{2}j$. When u is negative the relation $R(\sigma x)\leqq\lambda_1$ (which is satisfied when x is in $\overline{V}$) may be written in the form $\sigma_0 u-\sigma_1 v\leqq\lambda_1$ and obviously leads to the relation

$$-u\leqq\left|\frac{\sigma_1}{\sigma_0}\right||v|+\left|\frac{\lambda_1}{\sigma_0}\right|,$$

since σ_0 is negative. If $u<-\frac{1}{2}j$ we have

$$|v|>\tfrac{1}{2}\left|\frac{\sigma_0}{\sigma_1}\right|j-\left|\frac{\lambda_1}{\sigma_1}\right|,$$

whence we conclude at once to the desired result when $u<-\frac{1}{2}j$. This completes the proof.

as polynomials in ρ, where ρ is defined as in the theorem. It is clear that the highest power of ρ in the coefficient of each power of j has a coefficient whose real part is negative, since

$$\sigma_0 \cos s\theta - \sigma_1 \sin s\theta \leqq -\varepsilon$$

for each value of s in the set $s=0, 1, 2, \ldots, t$. Now, for any given positive number K_1, however large, it is clear that a constant X_1 exists such that $|x+j| > e^{K_1}$ for every j and every x such that $|x| \geqq X_1$. Hence, if we replace $|x+j|$ in (13) by e^{K_1}, having properly chosen K_1 as a quantity independent of x and j, and write the resulting relation in the form

$$|\overline{E}| < e^{\pi(\rho, j)},$$

it is clear that $\pi(\rho, j)$ is a polynomial in j whose coefficients are all negative provided that ρ is greater than an appropriately chosen constant X_2. Hence $|\overline{E}| < 1$ for every j provided that $|x| \geqq X_2$. Hence the expression in (12) has the desired property of being less than 1 in absolute value for every j greater than s_1, provided only that $|x|$ is greater than an appropriately chosen constant X.

In the case in which $k < m$ the same conclusion may readily be obtained in a similar manner. In this case the polynomial $Q_1(x+j)$ plays the leading rôle and in a corresponding way, since the leading term in $Q_1(x)$, considered as a polynomial in x, is $m\sigma x^{m-1}$. In carrying out the argument it is convenient to replace $|x+j|$ by $e^{K_2(j+|u|+|v|)}$ and to observe that the latter quantity is greater than the former, however small the constant K_2 may be, provided only that $|x|$ is sufficiently large. It is easy to supply the detailed argumentation requisite here.

This completes the proof of the theorem for the case in which k or m is greater than unity.

For the case which remains we have $k=1$ and $m=0$ or 1.* Moreover, we have seen that it is sufficient to know that the series $T(x, s)$ is bounded for x approaching infinity in $\overline{V}$. But the sum of any finite number of terms of $T(x, s)$ approaches a finite value as x thus approaches infinity. Omitting $r-1$ terms of $T(x, s)$ we see that it is sufficient to our purpose to know that the quantity

$$\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+1)h(x+n)}{h(x+s+1)h(x_0+n)} \right| \cdot \left| \frac{g(x+n+1)g(x_0+n)}{g(x+n)g(x_0+n+1)} - 1 \right| \tag{14}$$

is bounded as x approaches infinity in $\overline{V}$. The positive integer r which appears in this expression may be chosen to suit our convenience.

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* If λ_1 were chosen so as to be less than $\lambda + 1/R(\sigma)$ and the definition of $\overline{V}$ were modified accordingly, the theorem could be proved in this case just as in the preceding. But for the stronger form of the theorem a modified argument is necessary.

For the case now in consideration $h(x)$ has the special value

$$h(x) = x^{\mu_0+\sigma x} e^{\alpha_0+\beta x}.$$

Then, from the relation between $h(x)$ and $g(x)$, it follows that $g(x+1)/g(x)$ may be written in the form

$$\frac{g(x+1)}{g(x)} = x^{\sigma} e^{\sigma+\beta}\left(1+\frac{\xi(x)}{x}\right), \tag{15}$$

where the function $\xi(x)$ is bounded as x approaches infinity in $\overline{V}$. By means of this relation we readily have

$$\frac{g(x+n+1)\,g(x_0+n)}{g(x+n)\,g(x_0+n+1)} - 1 = \frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}} - 1 + \frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}} \cdot \frac{r(x+n)-r(x_0+n)}{1+r(x_0+n)},$$

where $r(x) = \xi(x)/x$. From these relations and the fact that $|(x+n)^{\sigma}/(x_0+n)^{\sigma}|$ is bounded it is easy to see that the quantity in (14) is bounded provided that the same is true of each of the quantities

$$\left.\begin{aligned} &\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+1)\,h(x+n)}{h(x+s+1)\,h(x_0+n)} \right| \cdot \left| \frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}} - 1 \right|, \\ &\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+1)\,h(x+n)}{h(x+s+1)\,h(x_0+n)} \right| \cdot \frac{1}{n}. \end{aligned}\right\} \tag{16}$$

That the latter of these two expressions is bounded may be proved by the method employed in the preceding case in the derivation of a similar result. Hence it remains to be shown that the first expression in (16) is bounded as x approaches infinity in $\overline{V}$.

Now the first series in (16) may be written in the form

$$\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+r)\,h(x+n)}{h(x+s+r)\,h(x_0+n)} \right| \cdot \left| \frac{h(x_0+s+1)\,h(x+s+r)}{h(x+s+1)\,h(x_0+s+r)} \right| \cdot \left| \frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}} - 1 \right|.$$

Then from the form of $h(x)$ it is easy to see that the sum of this series is bounded provided that the same is true of the sum of the series

$$\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+r)\,h(x+n)}{h(x+s+r)\,h(x_0+n)} \right| \cdot |S(x,n)|, \tag{17}$$

where $S(x,n)$ denotes the quantity

$$S(x,n) = (x-x_0)^{(r-1)\sigma}\left\{ \frac{(x+n)^{\sigma}}{(x_0+n)^{\sigma}} - 1 \right\} = (x-x_0)^{(r-1)\sigma}\left\{ \left(1+\frac{x-x_0}{x_0+n}\right)^{\sigma} - 1 \right\}.$$

Now, if $|x-x_0| < |x_0+n|$ and x is in $\overline{V}$ it is easy to see through the application of the binomial theorem to the last expression for $S(x,n)$ that a constant M_1 exists such that

$$|S(x,n)| < \frac{M_1}{n}.$$

Again, the quantity enclosed in braces in the first expression for $S(x, n)$ is bounded, however x and n may vary provided only that $|x|$ is sufficiently large, since under such conditions $|x+n|/|x_0+n|$ is bounded away from zero (as we saw above). Hence, if $|x-x_0|\geqq|x_0+n|$ and x is in $\overline{V}$, it follows readily that a constant M_2 exists such that

$$|S(x, n)| < \frac{M_2}{n},$$

provided that the integer r is such that $r-1\geqq -1/R(\sigma)$. We choose r in such manner as to satisfy this relation.

Combining these two inequalities for $|S(x, n)|$ we see that a constant M_3 exists such that $|S(x, n)| < M_3/n$ for every x and n provided only that n and $|x|$ are sufficiently large and x is in $\overline{V}$. Hence the sum of the series in (17) is bounded provided that the same is true of the sum of the series

$$\sum_{n=s+r}^{\infty} \left| \frac{h(x_0+s+r)h(x+n)}{h(x+s+r)h(x_0+n)} \right| \cdot \frac{1}{n}.$$

That this sum is bounded may be proved by the method employed in the preceding case in the derivation of a similar result.

This completes the proof of Theorem I.

By $\omega(x)$ and $\overline{\omega}(x)$ we denote the series

$$\omega(x) = \sum_{n=0}^{\infty} c_n g_1(x+n), \tag{18}$$

$$\overline{\omega}(x) = \sum_{n=0}^{\infty} c_n \frac{g_1(x+n)}{g_1(x)}, \tag{18'}$$

where $g_1(x)$ is that special case of $g(x)$ in which $k=1$ and $m=0$ or 1. For the special case $\overline{\omega}(x)$ of the series $\overline{\Omega}(x)$ the foregoing theorem may readily be put in the following interesting form:

THEOREM II. *Let $\overline{\omega}(x)$ be a function defined by a series (18′) whose convergence number λ is not $-\infty$. Let $\overline{V}$ denote the greatest region of the x-plane common to V and the half-plane $R(\sigma x)\leqq\lambda_1$, λ_1 being a constant which is less than λ. Then, if x approaches infinity in $\overline{V}$, we have*

$$\lim x^{-s\sigma}\left\{\overline{\omega}(x) - \sum_{n=0}^{s} c_n \frac{g_1(x+n)}{g_1(x)}\right\}=0, \qquad s=0, 1, 2, \ldots\ldots \tag{19}$$

For the proof of the theorem it is sufficient to observe that

$$\frac{g_1(x+s)}{x^{s\sigma}g_1(x)}$$

approaches a finite quantity as x approaches infinity in $\overline{V}$, a result which follows readily from (15), or directly from the asymptotic form of $g_1(x)$.

For an important range of cases the region $\overline{V}$ in Theorem II may be replaced by the half-plane $R(\sigma x) \leqq \lambda_1$, namely, those cases in which the rays bounding the sector V (except for their common point zero) lie entirely within the half-plane $R(\sigma x) \geqq 0$. In these cases relations (19) are valid for x approaching infinity in any half-plane lying within the half-plane of convergence of the series $\overline{\omega}(x)$. In particular, this is the case when $g_1(x)$ is the first principal solution of the difference equation

$$f(x+1) = A(x) f(x),$$

in which $A(x)$ is a function which is analytic at infinity and has there a zero of the first order. The series $\overline{\omega}(x)$, defined by means of such a function $g_1(x)$, are of prime importance as we shall show in a later paper. For the special case when $A(x) = 1/x$ the series $\overline{\omega}(x)$ is the factorial series. Theorem II for the special case of factorial series is due to Nörlund.* The demonstration employed by Nörlund in the special case does not seem to be applicable for deriving the general theorems here obtained.

§ 2. *Uniformity of Convergence of the Series $\overline{\Omega}(x)$.*

From relation (7) it follows that constants N_1 and N_2 exist such that

$$\left.\begin{aligned} \left| \sum_{n=s+1}^{\infty} c_n \frac{g(x+n)}{g(x)} \right| &\leqq N_1 \sum_{n=s+1}^{\infty} \left| \frac{g(x+n)}{g(x) g(x_0+n)} \right| \cdot \left| \frac{g(x+n+1) g(x_0+n)}{g(x+n) g(x_0+n+1)} - 1 \right| \\ &\leqq N_2 \sum_{n=s+1}^{\infty} \left| \frac{h(x+n) h(x_0+r)}{h(x+r) h(x_0+n)} \right| \cdot \left| \frac{h(x+r) h(x_0)}{h(x) h(x_0+r)} \right| \\ &\qquad \cdot \left| \frac{g(x+n+1) g(x_0+n)}{g(x+n) g(x_0+n+1)} - 1 \right| \end{aligned}\right\} \quad (20)$$

for every integer s greater than 1, r being a positive integer less than s. When $k=1$ and $m=0$ or 1, one may show, by the method employed in the latter part of the proof of Theorem I in the preceding paragraph, that a constant N_3 exists such that

$$\left| \sum_{n=s+1}^{\infty} c_n \frac{g(x+n)}{g(x)} \right| \leqq N_3 \sum_{n=s+1}^{\infty} \left| \frac{h(x+n) h(x_0+r)}{h(x+r) h(x_0+n)} \right| \cdot \frac{1}{n}, \qquad (21)$$

provided that $r > 1/R(\sigma)$ and s is greater than r.

Now, let x_1 be a non-exceptional point for the series $\overline{\Omega}(x)$ such that $\lambda_1 < R(\sigma x_1) < R(\sigma x_0)$. Denote by (20_1) the relation obtained from (20) by replacing the last expression in absolute value signs by the sum of the absolute values of its terms. Then the series in (20_1) dominates that in (20). But if either k or m is greater than unity the series in (20_1) converges when x is

* *Acta Mathematica*, Vol. XXXVII (1914), pp. 327–387.

replaced by x_1 and this resulting series dominates (20_1) itself, at least if $|x|$ is sufficiently large and x is in $\overline{V}$, as one sees through aid of the fact that the quantity in (12) is less than 1 in absolute value when $|x|$ is sufficiently large. If $k=1$ and $m=0$ or 1, the series in (21) converges when x is replaced by x_1, and as before, the resulting series dominates that in (21) itself, at least if $|x|$ is sufficiently large and x is in $\overline{V}$. Hence, in either case, there exists a convergent series of constant terms $\gamma_1+\gamma_2+\gamma_3+\ldots.$, such that we have the term by term inequality

$$\sum_{n=s+1}^{\infty}\left|c_n\frac{g(x+n)}{g(x)}\right|<\gamma_{s+1}+\gamma_{s+2}+\ldots., \tag{22}$$

provided that $s>1$ and $s>r+1$, at least if $|x|$ is sufficiently large and x is in $\overline{V}$.

Now, in Theorem III of *I*, we saw that $\overline{\Omega}(x)$ converges uniformly in any closed domain D which lies within its region of convergence and contains no points which are exceptional for this series, or are limit points of points which are exceptional for this series. From this result and relation (22), we conclude at once to the following theorem:

THEOREM III. *Let S be any region which lies in $\overline{V}$* and does not contain either in its interior or on its boundary a point which is exceptional for the series $\overline{\Omega}(x)$ or is a limit point of points which are exceptional for the series $\overline{\Omega}(x)$. The series $\overline{\Omega}(x)$ converges uniformly in S.*

This theorem for the special case of factorial series is due to Nörlund (*l. c.*). Nörlund's method of demonstration in the special case does not seem to be applicable for deriving the general theorem here obtained.

§ 3. *Extension of the Notion of Asymptotic Representation.*

In the complex x-plane let D be any region which extends to infinity, as for instance a sector bounded by two rays from zero to infinity. A function $f(x)$ is said to have in D the Poincaré asymptotic power series representation

$$f(x)\sim c_0+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots., \tag{23}$$

provided that

$$\lim x^s\left\{f(x)-\left(c_0+\frac{c_1}{x}+\ldots.+\frac{c_s}{x^s}\right)\right\}=0, \qquad s=0,1,2,\ldots.,$$

the limit in each case being taken for x approaching infinity in D. The power series in (23) may be either convergent or divergent.

* For the definition of the region $\overline{V}$ see Theorem I.

In analogy with this definition and guided by the result contained in Theorem I, it is natural to introduce an extended notion of asymptotic representation. Accordingly, we shall say that a function $f(x)$ has in D the asymptotic Ω-representation

$$f(x) \sim \sum_{n=0}^{\infty} c_n g(x+n), \tag{24}$$

with respect to $g(x)$ provided that

$$\lim \{g(x+s)\}^{-1}\{f(x) - \sum_{n=0}^{s} c_n g(x+n)\} = 0, \qquad s=0, 1, 2, \ldots,$$

the limit in each case being taken for x approaching infinity in D. Similarly we shall say that a function $f(x)$ has in D the asymptotic $\bar{\Omega}$-representation

$$f(x) \sim \sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}, \tag{25}$$

with respect to $g(x)$ provided that

$$\lim \frac{g(x)}{g(x+s)}\left\{f(x) - \sum_{n=0}^{s} c_n \frac{g(x+n)}{g(x)}\right\} = 0, \qquad s=0, 1, 2, \ldots,$$

the limit being taken as before. For the purpose of these definitions it is not necessary that the series in (24) and (25) shall converge in D; in fact, they may be divergent for all values of x.

That the foregoing definition of asymptotic $\bar{\Omega}$-representation includes the asymptotic power series representation of Poincaré, as a special case may be seen by giving to $g(x)$ the value e^{-x^2}, and in the resulting asymptotic series (25) replacing e^{2x} by z; for the series in (25) then becomes a descending power series in z, and the factor $g(x)/g(x+s)$ is equal to the product of z^s by a quantity which is independent of z and different from zero. But this relation between the two types of asymptotic representation is relatively unimportant when compared with that which is brought to light in the next section.

It is clear that the asymptotic relations (24) and (25) are most delicate when k and m, the degrees of the polynomials $P(x)$ and $Q(x)$, are least. Consequently, it is natural to treat first the special case in which $k=1$ and $m=0$ or 1. It will indeed be seen that this case is of leading importance in the theory. Moreover, it is the case which is associated in the most valuable way with the notion of asymptotic representation from the point of view of Poincaré. Accordingly, we devote the next three sections to a treatment of this case.

§ 4. *On a Special Case of Asymptotic $\bar{\Omega}$-Representation in Relation to the Asymptotic Representation of Poincaré.*

In what follows we shall assume that the region D defined in § 3 lies in V. In the special case when $g(x)$ has the value $g_1(x)$, where

$$g_1(x) \sim x^{\mu_0+\sigma x} e^{a_0+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots.\right), \tag{26}$$

the limits defining the asymptotic relation

$$f(x) \sim \sum_{n=0}^{\infty} c_n \frac{g_1(x+n)}{g_1(x)} \tag{27}$$

are equivalent to the limits

$$\lim x^{-s\sigma}\left\{f(x)-\sum_{n=0}^{s} c_n \frac{g_1(x+n)}{g_1(x)}\right\}=0, \qquad s=0,1,2,\ldots., \tag{28}$$

the limit in each case being taken for x approaching infinity in D. This follows readily from the fact that $g_1(x+s)x^{-s\sigma}/g(x)$ approaches a finite non-zero quantity as x approaches infinity in V.

We shall now obtain another set of limits equivalent to those in (28). For this purpose we observe that we have a Poincaré asymptotic representation of the form

$$\frac{g_1(x+n)}{g_1(x)} \sim x^{n\sigma}\left(\alpha_{0n}+\frac{\alpha_{1n}}{x}+\frac{\alpha_{2n}}{x^2}+\ldots.\right),$$

where $\alpha_{0n}, \alpha_{1n}, \ldots.$ are a set of numbers dependent only on n and the constants which enter into the asymptotic formula (26) for $g_1(x)$. Thence, we see that relations (28) are equivalent to the following:

$$\lim x^{-s\sigma}\left\{f(x)-\sum_{n=0}^{s} c_n x^{n\sigma}\left(\alpha_{0n}+\frac{\alpha_{1n}}{x}+\ldots.+\frac{\alpha_{n_s n}}{x^{n_s}}\right)\right\}=0, \qquad s=0,1,2,\ldots.. \tag{29}$$

Here the quantities n_s are integers dependent only on n and s. It is clear that a suitable value of n_s is any integer not less than $(s-n)(|\sigma_0|+|\sigma_1|)$. In particular, if $\sigma=-1$ we may take $n_s=s-n$; and this we do.

For the special case in which $\sigma=-1$ it is easy to see that relations (29) are equivalent to the Poincaré asymptotic formula

$$f(x) \sim \beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\ldots., \tag{30}$$

where

$$\beta_\nu = c_0\alpha_{\nu 0}+c_1\alpha_{\nu-1,1}+c_2\alpha_{\nu-2,2}+\ldots.+c_\nu\alpha_{0\nu}.$$

Hence, relation (27) with $\sigma=-1$ gives rise to the Poincaré asymptotic representation (30). In the next section we shall show that these two relations are

indeed equivalent, and shall exhibit the explicit formulae which put in evidence this equivalence. Thus we shall see that relation (27) contains (30) as a special case and shall make clear the importance of the former in view of the fundamental character of the latter.

Again, if we give to σ the rational value $-p/q$, where p and q are relatively prime positive integers, it is clear that relations (29) lead to that generalization of (30) in which x is replaced by a q-*th* root of x. In general there are necessary relations among the resulting quantities β; but if $p=1$ it is easy to see that they may have any values whatever when the quantities $c_0, c_1, c_2, \ldots$ are arbitrary.

If we give to σ an irrational real value and derive the formula corresponding to (30), we shall find that the powers of x which enter into it are not only not of integral index, but also that these indices are not integral multiples of any number whatever. This brings out the fact that (27) affords an essential generalization of (30).

On account of the leading importance of the case in which $\sigma=-1$, this case alone will be treated in the remainder of the paper.

§ 5. *Explicit Formulae Connecting Relations* (27) *and* (30) *when* $\sigma=-1$.

In the special case when $\sigma=-1$ we shall denote $g_1(x)$ by $\bar{g}(x)$, so that we have

$$\bar{g}(x) \sim x^{\mu_0-x} e^{\alpha_0+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\right). \tag{31}$$

We have seen that a relation of the form

$$\frac{\bar{g}(x+n)}{\bar{g}(x)} \sim x^{-n}\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\frac{\beta_{2n}}{x^2}+\ldots\right) \tag{32}$$

exists, where $\beta_{0n}, \beta_{1n}, \ldots$ are a set of numbers independent of x. We shall now determine the values of these numbers β_{ij} in terms of $\mu_0, \alpha_0, \beta, a_1, a_2, \ldots$

Setting

$$\bar{h}(x) = x^{\mu_0-x} e^{\alpha_0+\beta x},$$

we have

$$\frac{\bar{h}(x+n)}{\bar{h}(x)} = x^{-n} e^{(\beta-1)n}\left(1+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\ldots\right), \tag{33}$$

where the constants $\gamma_{1n}, \gamma_{2n}, \ldots$ are readily determined consecutively by expanding the function in the second member of the relation

$$1+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\ldots = \left(1+\frac{n}{x}\right)^{\mu_0-n} e^{\frac{n^2}{2x}-\frac{n^3}{3x^2}+\frac{n^4}{4x^3}-\cdots}$$

If we multiply (32) through by $\bar{g}(x)$, replace $\bar{g}(x+n)$ and $\bar{g}(x)$ by their asymptotic values from (31) and reduce the result by means of (33) we readily obtain the following relation:

$$e^{(\beta-1)n}\left(1+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\dots\right)\left(1+\frac{a_1}{x+n}+\frac{a_2}{(x+n)^2}+\dots\right)$$
$$\sim\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\dots\right)\left(1+\frac{a_1}{x}+\dots\right). \quad (34)$$

But

$$(x+n)^{-k}=x^{-k}\left(1+\frac{n}{x}\right)^{-k}$$
$$=x^{-k}\left(1-\frac{k}{1!}\cdot\frac{n}{x}+\frac{k(k+1)}{2!}\cdot\frac{n^2}{x^2}-\frac{k(k+1)(k+2)}{3!}\cdot\frac{n^3}{x^3}+\dots\right);$$

whence we have

$$1+\frac{a_1}{x+n}+\frac{a_2}{(x+n)^2}+\dots\sim 1+\frac{\bar{a}_1}{x}+\frac{\bar{a}_2}{x^2}+\dots \quad (35)$$

where

$$\bar{a}_k=a_k-(k-1)na_{k-1}$$
$$+\frac{(k-1)(k-2)}{2!}n^2a_{k-2}-\frac{(k-1)(k-2)(k-3)}{3!}n^3a_{k-3}+\dots\dots \quad (36)$$

Replacing the second series in (34) by its value from (35) and equating coefficients of like powers of x we have the following recurrence relations for determining $\beta_{0n}, \beta_{1n}, \dots$:

$$\left.\begin{aligned}\beta_{0n}&=e^{(\beta-1)n},\\ \beta_{1n}+a_1\beta_{0n}&=e^{(\beta-1)n}(\bar{a}_1+\gamma_{1n}),\\ &\dots\dots\dots,\\ \sum_{i=0}^{k}a_i\beta_{k-i,n}&=e^{(\beta-1)n}\sum_{i=0}^{k}\bar{a}_{k-i}\gamma_{in},\end{aligned}\right\} \quad (37)$$

where $a_0=c_0=\bar{a}_0=1$. It is evident that these relations uniquely determine the quantities $\beta_{0n}, \beta_{1n}, \dots$.

From this it follows readily that the asymptotic $\bar{\Omega}$-representation

$$f(x)\sim\sum_{n=0}^{\infty}c_n\frac{\bar{g}(x+n)}{\bar{g}(x)} \quad (38)$$

leads to the Poincaré asymptotic representation

$$f(x)\sim\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\dots, \quad (39)$$

where

$$\beta_\nu=c_0\beta_{\nu 0}+c_1\beta_{\nu-1,1}+c_2\beta_{\nu-2,2}+\dots+c_\nu\beta_{0\nu}. \quad (40)$$

On the other hand, if (39) is given then from (40) the constants $c_0, c_1, c_2, \dots$ may be uniquely determined (since $\beta_{0\nu}\neq 0$); and hence, conversely, relation (39) leads to relation (38).

This is sufficient to establish the complete equivalence between the Poincaré asymptotic relation (39) and the asymptotic $\bar{\Omega}$-representation (38). Later we shall point out an important advantage which the latter has over the former in that the series in (38) is convergent (in a half-plane) for a large class of functions $f(x)$ for which the corresponding series in (39) is everywhere divergent.

In an important class of cases the asymptotic character of the function $\rho(x)$,

$$\rho(x)=\frac{\bar{g}(x+1)}{\bar{g}(x)},$$

is known directly and has a simple form. (It can of course be computed in the general case.) In case this asymptotic series for $\rho(x)$ is convergent, and actually represents $\rho(x)$, one has readily the descending power series expansion of $\rho'(x)/\rho(x)$. It is easy to see that it has the form

$$\frac{\rho'(x)}{\rho(x)}=-\left(\frac{1}{x}+\frac{\mu_{20}}{x^2}+\frac{\mu_{30}}{x^3}+\ldots.\right), \tag{41}$$

where $\mu_{20}, \mu_{30}, \ldots.$ are constants depending on the constants in $\bar{g}(x)$. In the important special case in which $\bar{g}(x)=1/\Gamma(x)$ it is easy to see that (41) takes the simple form $\rho'(x)/\rho(x)=-1/x$.

The constants β_{in} may be determined readily in terms of the constants $\mu_{20}, \mu_{30}, \ldots.$; and the results thus obtained will sometimes be in more convenient form than those obtained by solving equations (37) for β_{in}. For effecting this determination let us write

$$y_n\equiv\frac{\bar{g}(x+n)}{\bar{g}(x)}=\rho(x)\rho(x+1)\ldots.\rho(x+n-1);$$

whence we have

$$\frac{y_n'}{y_n}=\frac{\rho'(x)}{\rho(x)}+\frac{\rho'(x+1)}{\rho(x+1)}+\ldots.+\frac{\rho'(x+n-1)}{\rho(x+n-1)}.$$

From (41) it is clear that we have an expansion of the form

$$\frac{\rho'(x+j)}{\rho(x+j)}=-\left(\frac{1}{x}+\frac{\mu_{2j}}{x^2}+\frac{\mu_{3j}}{x^3}+\ldots.\right).$$

Moreover, since

$$(x+j)^{-l}=x^{-l}\left(1-\frac{l}{1!}\cdot\frac{j}{x}+\frac{l(l+1)}{2!}\cdot\frac{j^2}{x^2}-\frac{l(l+1)(l+2)}{3!}\cdot\frac{j^3}{x^3}+\ldots.\right),$$

it is easy to see that μ_{ij} has the value

$$\mu_{ij}=\mu_{i0}-j\mu_{i-1,0}\frac{i-1}{1!}+j^2\mu_{i-2,0}\frac{(i-1)(i-2)}{2!}-\ldots.$$
$$+(-1)^{i-2}j^{i-2}\mu_{20}(i-1)+(-1)^{i-1}j^{i-1}. \tag{42}$$

Hence we have

$$\frac{y'_n}{y_n} = -\left(\frac{n}{x} + \frac{\varepsilon_{1n}}{x^2} + \frac{2\varepsilon_{2n}}{x^3} + \frac{3\varepsilon_{3n}}{x^4} + \ldots\right),$$

where

$$i\varepsilon_{in} = \sum_{j=0}^{n-1} \mu_{i+1,\,j}. \tag{43}$$

Therefore,

$$\frac{\bar{g}(x+n)}{\bar{g}(x)} \equiv y_n = x^{-n}\beta_{0n} e^{\frac{\varepsilon_{1n}}{x} + \frac{\varepsilon_{2n}}{x^2} + \ldots\ldots}$$

By comparison with (32) we have

$$\frac{1}{\beta_{0n}}\left(\beta_{0n} + \frac{\beta_{1n}}{x} + \frac{\beta_{2n}}{x^2} + \ldots\right) = 1 + \left(\frac{\varepsilon_{1n}}{x\,!} + \frac{\varepsilon_{2n}}{x^2} + \ldots\right)$$
$$+ \frac{1}{2\,!}\left(\frac{\varepsilon_{1n}}{x} + \frac{\varepsilon_{2n}}{x^2} + \ldots\right)^2 + \ldots\ldots$$

Thence we see that

$$\frac{\beta_{in}}{\beta_{0n}} = 1 + \varepsilon_{in} + \frac{1}{2\,!}\Sigma\varepsilon_{j_1n}\varepsilon_{j_2n} + \frac{1}{3\,!}\Sigma\varepsilon_{j_1n}\varepsilon_{j_2n}\varepsilon_{j_3n} + \ldots, \tag{44}$$

where in each case the summation is to be taken for varying subscripts j such that their sum in each case shall be i.

Relations (42), (43), (44) serve to express β_{in} in terms of $\mu_{20}, \mu_{30}, \ldots\ldots$ Thence through (40) we have the relation existing between the constants in the asymptotic formulae (38) and (39).

In case $\rho(x)$ is a rational function the constants β_{in} may be determined in a very simple manner. For the sake of simplicity in the formulae we shall assume that the poles of $\rho(x)$ are of the first order, and that no two of its polar points differ in affixes by an integer. Denoting the poles of $\rho(x)$ by $p_1, p_2, \ldots, p_h$ we have a partial fraction expansion of the form

$$y_n \equiv \rho(x)\rho(x+1)\ldots\rho(x+n-1) = \sum_{l=1}^{h}\sum_{j=0}^{n-1}\frac{\eta_{lj}}{x-p_l+j}\,; \tag{45}$$

whence we see readily that

$$y_n = \sum_{l=1}^{h}\sum_{j=0}^{n-1}\eta_{lj}\left(\frac{1}{x} + \frac{p_l-j}{x^2} + \frac{(p_l-j)^2}{x^3} + \ldots\right).$$

Therefore,

$$\beta_{in} = \sum_{l=1}^{h}\sum_{j=0}^{n-1}\eta_{lj}(p_l-j)^{n+i-1}. \tag{46}$$

As an example take the case in which $\rho(x) = 1/x$. Then we have

$$\frac{1}{x(x+1)\ldots(x+n-1)} = \frac{1}{x} - \frac{1}{1\,!}\,\frac{1}{x+1}$$
$$+ \frac{1}{2\,!}\,\frac{1}{x+2} - \ldots + (-1)^{n-1}\frac{1}{(n-1)\,!}\,\frac{1}{x-n+1},$$

and

$$\beta_{in} = (-1)^{n+i-1}\sum_{j=0}^{n-1}(-1)^j\frac{1}{j\,!}\,j^{n+i-1}. \tag{47}$$

That the two asymptotic formulae (38) and (39) are equivalent in the special case in which $\rho(x)=1/x$ has been shown by Nielsen,* who obtained formulae equivalent to (47) and (40) for the expression of this equivalence.

§ 6. *Illustrative Example.*

Let us consider the function $f(x)$ defined by the series

$$f(x)=\sum_{n=0}^{\infty} c_n \frac{\bar{g}(x+n)}{\bar{g}(x)}, \tag{48}$$

where the constants c_n are subject to the condition

$$|c_n|<M(n!)^{1-\epsilon},$$

where M and ϵ are positive constants. This series is term by term less in absolute value than the series

$$\sum_{n=0}^{\infty} M(n!)^{1-\epsilon}\left|\frac{\bar{g}(x+n)}{\bar{g}(x)}\right|.$$

That the latter series converges for every non-exceptional value of x is readily shown by the Cauchy ratio test. Hence the series in (48) converges for every non-exceptional value of x. Therefore, it defines a function $f(x)$ which is analytic at every point x which is non-exceptional for the series in (48). At the exceptional points for this series $f(x)$ will in general have singularities. In particular, if $\rho(x)$ is a rational function such that its poles are of the first order and no two of its polar points differ in affix by an integer, then $f(x)$ in general will have an infinite number of poles of the first order having infinity as their sole limit point. In such a case as this the power series in (39) diverges for every value of x. It yields the asymptotic character of $f(x)$, but does not yield anything else directly. On the other hand, the series in (48) also yields the asymptotic character of $f(x)$ and—what is much more—it affords a convergent representation of $f(x)$ at all points which are non-exceptional for the series in (48); and these latter points are in general points of singularity of $f(x)$. Moreover, the series furnishes in general a ready means for investigating the character of the singularities of $f(x)$.

This example illustrates an important result which I expect to present in full in a subsequent paper. It turns out that important classes of functions exist, each function of which has an essential singularity at infinity, but nevertheless admits a convergent $\bar{\Omega}$-expansion. Moreover, as one sees from the present paper, such an $\bar{\Omega}$-expansion exhibits directly those properties of the function which in many investigations one desires first of all to derive.

University of Illinois, *April*, 1916.

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* Nielsen, *Annales Scientifiques de l'École Normale Supérieure*, Series 3, Vol. XXI (1904), pp. 449–458.

Possible Characteristic Operators of a Group.

By G. A. Miller.

§ 1. *Introduction.*

A characteristic operator of a group G is an operator which corresponds to itself in every possible automorphism of G. Every possible group contains at least one characteristic operator, viz., the identity. A necessary and sufficient condition that an abelian group involves another characteristic operator is that it contains one and only one operator of order 2. Hence we may confine our attention to non-abelian groups in the study of possible characteristic operators. In what follows we shall therefore assume that G is non-abelian.

The characteristic operators of G clearly constitute an abelian characteristic subgroup of G. The main object of the present article is to prove that it is possible to construct a non-abelian group which has for its characteristic operators all the operators of an arbitrary abelian group. By means of the fact that at least one such non-abelian group can be constructed, it is easy to prove that the number of such groups is always infinite. In fact, such an infinite system can be obtained directly by forming the direct products of one such group and simple groups of composite orders.

§ 2. *Group Containing a Characteristic Operator Whose Order is an Arbitrary Power of a Prime Number.*

Let p represent any prime number of the form $1+kq^{\alpha}$, α being an arbitrary positive integer. The transitive substitution group K of degree p and of order $p(p-1)$ contains a subgroup of order pq^{α} which involves no invariant operator except identity. To obtain the desired group in the form of a regular substitution group, we shall first construct an intransitive substitution group H by establishing a (1, 1) correspondence between q^{α} regular cyclic groups of order pq^{β}, $\beta \overline{\gtrless} \alpha$. The q^{α} transitive constituents of H are conjugate under the powers of a substitution t of order q^{α} which is commutative with every substitution of H, q being a prime number.

Let s be a substitution of order q^{α} which is commutative with t and with the substitutions of order q^{β} contained in H, but transforms the substitutions of order p contained in H into a power which belongs to exponent q^{α} mod p. That is, the group generated by s and a substitution of order p contained in H is simply isomorphic with K, and we may assume that the group generated by s and H involves q^{α} transitive constituents on the letters of H. The product of st and a substitution of order q^{β} in one of the transitive constituents of H is of order $q^{\alpha+\beta}$, and the q^{α}-*th* power of this product is the substitution of order q^{β} contained in H.

The group G generated by H and this product has for its central the cyclic group of order q^{β}, and the central quotient group of G is simply isomorphic with K. Since G involves only one subgroup of order p, and since all the operators of G which transform one of the generating operators of this subgroup into its n-*th* power must also transform each of its other operators into the same power, it results that in every possible automorphism of G the operators which transform its operators of order p into a particular power must correspond to themselves. In particular, the operators of order $q^{\alpha+\beta}$ in G which transform each of its operators of order p into the same power must correspond to themselves in each of these automorphisms.

The operators of order $q^{\alpha+\beta}$ in G whose q^{α}-*th* power is the same operator of order q^{β} in H clearly transform the operators of order p in H into one or more powers, which are distinct from the powers into which these operators of order p are transformed by those operators of order $q^{\alpha+\beta}$ whose q^{α}-*th* power is another operator of order q^{β} in H. From this it results directly that the operators of order q^{β} contained in the central of G are characteristic operators of G. Since these operators generate the central of G, this central must be composed of characteristic operators of G.

Since q is an arbitrary prime number, and β is an arbitrary positive integer, we have established the lemma that *it is possible to construct a non-abelian group which has an operator of any desired prime power order as a characteristic operator.* In fact, the existence of such a characteristic operator of order q^{β} is independent of p and α provided they satisfy the stated conditions, and hence it results from the preceding arguments that any one of a multiply infinite number of different groups may be used for G.

§ 3. *General Case.*

To prove that it is possible to construct a group G which has all the operators of an arbitrary abelian group G' as characteristic operators, it is

only necessary to prove that a set of independent generators of G' are characteristic operators of a certain group G. We shall first consider the case when the order of each of the independent generators of G' is a power of q, and we shall let q^β represent one of the largest of these orders.

Let $s_1, s_2, \ldots, s_r$ represent a set of independent generators of G' and find r distinct prime numbers $p_1, p_2, \ldots, p_r$ of the form $1+kq^\beta$. According to the preceding section we can construct r distinct groups $G_1, G_2, \ldots, G_r$ such that the characteristic operators of these groups are generated by $s_1, s_2, \ldots, s_r$, respectively, and that the prime numbers of the form $1+kq^\beta$ which divide their orders are $p_1, p_2, \ldots, p_r$, respectively. The orders of the r groups $G_1, G_2, \ldots, G_r$ may be assumed to be

$$p_1q^\beta, p_2q^\beta, \ldots, p_rq^\beta$$

multiplied by the orders of $s_1, s_2, \ldots, s_r$, respectively.

The direct product of $G_1, G_2, \ldots, G_r$ contains a characteristic cyclic subgroup of order $p_1, p_2, \ldots, p_r$. In any automorphism of this direct product G_1 corresponds to a subgroup whose operators are commutative with each of the operators in the cyclic subgroup of order $p_2, p_3, \ldots, p_r$ contained in this direct product. To the operators of order $q^{\alpha'+\beta}$ in G_1, α' being the order of s_1, there must therefore correspond operators whose q^β power is the same as the q^β power of the corresponding operators of G_1. Hence s_1 is a characteristic operator of this direct product. As s_1 is an arbitrary independent generator of G', the direct product under consideration has all the operators of G' for its characteristic operators.

When the order of G' is not a power of a prime number, G' is the direct product of characteristic subgroups whose orders are such powers. By constructing groups whose characteristic operators constitute these characteristic subgroups, and then forming the direct product of these groups, we obtain a group whose characteristic operators are the operators of G'. This completes a proof of the theorem

It is possible to construct non-abelian groups whose characteristic operators constitute an arbitrary abelian group.

*Linear Differential Equations in Infinitely Many Variables.**

BY WILLIAM L. HART.

§ 1. *Introduction.*

In the study of finite systems of ordinary differential equations, linear systems

$$\frac{dx_i}{dt}=\sum_{j=1}^{n} k_{ij}(t)x_j, \qquad (i=1, 2, \ldots, n), \tag{1}$$

are found to have many interesting properties, a number of which are connected with the notion of fundamental sets of solutions of (1). In the present paper the infinite system of ordinary differential equations

$$\frac{dx_i}{dt}=\sum_{j=1}^{\infty} k_{ij}(t)x_j, \qquad (i=1, 2, \ldots), \tag{2}$$

is considered and certain analogous properties are discussed where the notion of fundamental sets of solutions for (2) is defined in an appropriate manner.

Throughout the discussion it is assumed that the $k_{ij}(t)$ in (2) are power series in the complex variable t converging for $|t|\leq r$. The matrix of elements $(k_{ij}(t))_{i,j=1,2,\ldots}$ is supposed to be of the limited type, in the sense defined by Hilbert. The set of complex numbers $\xi=(x_1, x_2, \ldots)$ is supposed to represent a point in Hilbert complex space; *i. e.*, if $\bar{x}_i$ represents the conjugate to x_i, then $\Sigma_{i=1}^{\infty} \bar{x}_i x_i$ converges.

In § 2 the system of notation used in the paper is explained. There is then given a brief outline of the results from the theory of limited linear and bi-linear forms in infinitely many variables which are used in the later work. In § 3 the homogeneous system (2) is considered. Under certain assumptions as to the limited character of the matrix $(k_{ij}(t))$ the existence of an analytic solution $(x_1(t), x_2(t), \ldots)$ taking on, for $t=t_0$, a given set of values $(a_1, a_2, \ldots)$, is established and an analytical representation for it obtained in terms of an auxiliary matrix.

In § 4 the adjoint system to the equations (2),

$$\frac{dx_i}{dt}=-\sum_{j=1}^{\infty} k_{ji}(t)x_j, \tag{3}$$

* Presented to the American Mathematical Society, December 27, 1916.

is considered. As a result of the conditions satisfied by system (2), the system (3) is of the same type as (2). It is found that, if $\xi_1(t) = (x_{11}(t), x_{21}(t), \ldots.)$ and $\xi_2(t) = (x_{12}(t), x_{22}(t), \ldots.)$ are solutions of (2) and (3), respectively, then $\Sigma_{i=1}^{\infty} x_{i1}(t) x_{i2}(t)$ is a constant.

In § 5 the notion of fundamental sets of solutions of system (2) is defined in a manner suggested by that followed in the finite case if, in the latter, where there enters the idea of the non-vanishing of the determinant of the fundamental set, there is put the concept of a limited matrix possessing a unique limited reciprocal matrix. The existence of fundamental sets is established and certain of their properties are discussed. In this connection the adjoint system (3) proves to be useful because the reciprocal matrix of a fundamental set of solutions of (2) is a matrix of solutions of the system (3).

In § 6 the non-homogeneous system is considered which results from (2) on adding to the *i-th* equation the term $k_{i0}(t)$. The general solution of the modified system is obtained as the sum of a particular solution and the general solution of the system (2). On using a method analogous to that of variation of parameters, a particular solution is obtained by a quadrature with the aid of a fundamental set of solutions of (2).

Infinite systems of differential equations have been considered by H. Von Koch,* F. R. Moulton,† E. H. Moore,‡ J. F. Ritt,§ T. H. Hildebrandt,|| and the author.¶ The general systems of Von Koch, Moulton and the author do not include the system (2) of the present paper. J. F. Ritt considered a single differential equation of infinite order which is a problem of a different type. Moore and Hildebrandt considered problems in general analysis including as special instances a certain infinite system of the type (2), in the field of reals. The results of Moore were not of the nature of those obtained in the present paper. The results of Hildebrandt generalize the theorems on finite systems in such a way that the notion of the determinant of the fundamental sets of solutions is retained, whereas in the present paper the matrices of the fundamental sets need not possess determinants.

§ 2. *Limited Bi-Linear Forms.*

In the work of the present paper it will be convenient to adopt the system of notation which is explained in the following paragraphs. The same type of

* *Öfversigt af Kongliga Vetenskaps Akademiens Förhandlingar*, Vol. LVI (1899), pp. 395–411.
† *Proceedings of the National Academy of Sciences*, Vol. I (1915), p. 350.
‡ *Atti dei IV Congresso Internatzionale dei Matematici*, Vol. II (Roma, 1908), p. 98.
§ *Transactions of the American Mathematical Society*, Vol. XVIII (1917), p. 27.
|| *Ibid.*, Vol. XVIII (1917), p. 73.
¶ *Ibid.*, Vol. XVIII (1917), p. 125.

notation has long been used by E. H. Moore * in his work on Integral Equations. In its general form the notation is like that met in vector and matrix algebra.

An infinite set of values $(x_1, x_2, \ldots.)$ will be represented by a single Greek letter, for example ξ. Then, the point, or vector, ξ in the space of infinitely many dimensions is considered as a function of the variable i whose range is $I=(1, 2, \ldots.)$; that is, $\xi(i)=x_i$. Points $(x_1, x_2, \ldots.)$ will be represented either by ξ, η, γ or α, with perhaps an added notation.

An infinite matrix $(k_{ij})_{i,j=1,2,\ldots.}$ in which i denotes the row, j the column, will be represented by a single Greek letter such as $\varkappa$. Then $\varkappa$ is considered as a function of the two indices i and j for which $\varkappa(i, j)=k_{ij}$. Any Greek letter except ξ, η, γ or α will in the future represent a matrix.

A Greek letter with an added notation represents the point or matrix obtained by adding the given notation to each element of the point or matrix given by the unannotated Greek letter. For example,

$$\xi'=(x_1', x_2', \ldots.), \qquad \xi(t)=\{x_1(t), x_2(t), \ldots.\},$$
$$\varkappa(t)=(k_{ij}(t))_{i,j=1,2,\ldots.}, \qquad \varkappa_n=(k_{ijn})_{i,j=1,2,\ldots.},$$
$$\frac{d\xi}{dt}=\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots.\right), \qquad \frac{d\varkappa}{dt}=\left(\frac{dk_{ij}}{dt}\right)_{i,j=1,2,\ldots.}.$$

In the subsequent work infinite series enter to a great extent. On using what is seen to be analogous to vector notation, let there be introduced the abbreviation

$$\sum_{i=1}^{\infty} x_i y_i = S\xi\eta, \qquad [\xi=(x_1, x_2, \ldots.), \ \eta=(y_1, y_2, \ldots.)].$$

The S is a summation operation giving what may be called the scalar product of the two points or vectors ξ and η. In similar fashion, the point η whose coordinates y_i are given by $y_i=\Sigma_{j=1}^{\infty} k_{ij}x_j$ and the point α whose coordinates z_i are given by $z_i=\Sigma_{j=1}^{\infty} x_j k_{ji}$, will be represented by, respectively,

$$\eta=S\varkappa\xi, \quad \alpha=S\xi\varkappa.$$

The operation S thus produces a point from a matrix and a point. Let $\varkappa'$ and $\varkappa''$ be two matrices; the product matrix λ is defined by

$$\lambda=\left(\sum_{g=1}^{\infty} k_{ig}'k_{gj}''\right)_{i,j=1,2,\ldots.}=S\varkappa'\varkappa'',$$

or, S produces a matrix from two given matrices. The meaning of such expressions as $S\xi S\varkappa\eta$, $S\varkappa S\varkappa'\varkappa''$, $S(S\varkappa\xi)\eta$, is then evident. The expressions $S^2\xi\varkappa\eta$, $S^2\varkappa'\varkappa\xi$ are defined by, respectively,

$$S^2\xi\varkappa\eta=\sum_{i,j=1}^{\infty} x_i k_{ij} y_j = \lim_{n=\infty} \sum_{i,j=1}^{n} x_i k_{ij} y_j,$$
$$S^2\varkappa'\varkappa\xi=\left(\sum_{i,j=1}^{\infty} k_{1i}'k_{ij}x_j, \sum_{i,j=1}^{\infty} k_{2i}'k_{ij}x_j, \ldots.\right).$$

* *Proceedings of the Fifth International Congress* (Cambridge), Vol. I, p. 230.

In the following discussion, "$\Sigma_{i,j=1}^{\infty}$" will always mean the limit indicated in the first of the expressions above.

It will sometimes be convenient to use the expressions considered in the preceeding paragraph when the sums involved run only from 1 to m. Then write S_m instead of S:

$$S_m\xi\eta=\sum_{i=1}^{m} x_i y_i,\quad S_m x\xi=(\sum_{i=1}^{m} k_{1i}x_i,\ \sum_{i=1}^{m} k_{2i}x_i,\ \ldots\ldots),\ \text{etc.}$$

A stroke over a letter (*e. g.*, $\bar{x}$, $\bar{\xi}$) will always represent the conjugate complex number to the quantity bearing the notation.

DEFINITION 1. *A point ξ belongs to Hilbert complex space in case there exists*

$$S\bar{\xi}\xi=\sum_{i=1}^{\infty}\bar{x}_i x_i,\qquad [\xi=(x_1, x_2, \ldots\ldots)].$$

The number $\sqrt{S\bar{\xi}\xi}$ will be called the modulus of ξ and will be denoted by $M\xi$.

The proofs of certain of the properties of limited linear and bi-linear forms in infinitely many variables listed below are found in a paper* by E. Hellinger and O. Toeplitz. In the future the notation "*H. T. page* ——" refers to a page in this article. Hellinger and Toeplitz in most cases give the detailed proofs only for the case of real values of the variables, but as they remark (H. T. page 305), the extension to complex values is a simple matter.

PROPERTY 1 (*H. T. page* 295). *Suppose there exists a number l such that, for all points $\eta=(y_1, y_2, \ldots\ldots)$ of Hilbert space there holds, for every m,*

$$|S_m\xi\eta|\leqq lM\eta.$$

Then it follows that ξ belongs to Hilbert space and that $M\xi\leqq l$.

PROPERTY 2. *Let $\xi(u)$ and $\eta(u)$, where u is a complex variable belonging to a region R in the u-plane, belong to Hilbert space for every u. Suppose that*

$$S\bar{\xi}(u)\xi(u)\leqq l$$

for all u, while $S\bar{\eta}(u)\eta(u)$ converges uniformly for all u. Then

$$S\xi(u)\eta(u)=\sum_{i=1}^{\infty} x_i(u)y_i(u)$$

converges absolutely uniformly for all u of R.

This property is established with the aid of the Schwarz † inequality. For,

$$\sum_{i=n+1}^{\infty}|x_i(u)|\cdot|y_i(u)|\leqq\sqrt{\sum_{i=n+1}^{\infty}\bar{x}_i(u)x_i(u)}\sqrt{\sum_{i=n+1}^{\infty}\bar{y}_i(u)y_i(u)}\leqq l\sqrt{\sum_{i=n+1}^{\infty}\bar{y}_i(u)y_i(u)},$$

which approaches zero uniformly for all u as $n\doteq\infty$.

* *Mathematische Annalen*, Vol. LXIX (1910), p. 289.

† H. T. page 293.

DEFINITION 2. *A matrix* $\varkappa=(k_{ij})_{i,j=1,2,\ldots}$ *is limited* l *in case for every pair of Hilbert points* ξ *and* η *and for every* m,

$$|S_m S_m \xi\varkappa\eta| = |\sum_{i=1}^{m}\sum_{j=1}^{m} x_i k_{ij} y_j| \leqq lM\xi M\eta.$$

If $\varkappa$ is an infinite matrix, the conjugate matrix $\bar{\varkappa}$ and the transposed matrix $\varkappa'$ are defined by

$$\bar{\varkappa}=(\bar{k}_{ij})_{i,j=1,2,\ldots}, \quad \varkappa'=(k_{ji})_{i,j=1,2,\ldots}.$$

PROPERTY 3 (*H. T. page* 310). *Suppose that* $\varkappa$ *is limited* l. *Then* $\varkappa'$ *and* $\bar{\varkappa}$ *are also limited* l.

PROPERTY 4 (*H. T. page* 297). *Let* $\varkappa$ *be limited* l. *Then each row and each column of* $\varkappa$ *constitutes a point, belonging to Hilbert space, whose modulus is at most* l.

PROPERTY 5 (*H. T. page* 299). *If* $\varkappa$ *is limited* l *then, for every pair of Hilbert points* ξ *and* η, *the following infinite sums are defined and equal:*

$$S\xi S\varkappa\eta, \quad S(S\xi\varkappa)\eta, \quad S^2\xi\varkappa\eta.$$

PROPERTY 6 (*H. T. page* 300). *Let* $\varkappa'$ *and* $\varkappa''$ *be limited* l' *and* l'', *respectively. Then the product matrix* $\lambda=S\varkappa'\varkappa''$ *is limited* $l'l''$.

DEFINITION 3. *Let* $\varkappa$ *and* λ *be two limited matrices. The matrix* λ *is a right-hand, or a left-hand reciprocal of* $\varkappa$ *according as*

$$S\varkappa\lambda=\delta \quad \textit{or} \quad S\lambda\varkappa=\delta,$$

where δ *is the unit matrix* $(d_{ij})_{i,j=1,2,\ldots}$, $d_{ii}=1$, $d_{ij}=0\,(i\neq j)$.

PROPERTY 7 (*H. T. page* 311). *If the matrix* $\varkappa$ *is limited and possesses both right and left-hand reciprocals, it follows that there exists uniquely a right-hand and a left-hand reciprocal, and moreover, they are equal to each other.*

PROPERTY 8 (*H. T. page* 312). *Let* $\varkappa$ *be a limited matrix and suppose that there exists a unique right* (*left*) *hand reciprocal* λ. *Then* λ *is also the unique left* (*right*) *hand reciprocal of* $\varkappa$.

In case $\varkappa$ possesses a unique left-hand and right-hand reciprocal λ, this matrix will be called simply the reciprocal of $\varkappa$. By direct verification it is established that there holds.

PROPERTY 9. *Suppose that* $\varkappa$ *is limited and possesses a reciprocal* λ. *Then for every two points* ξ *and* η *of Hilbert space, the two infinite systems of linear equations*

$$\xi=S\varkappa\eta, \quad \eta=S\lambda\xi$$

are equivalent.

§3. *The Homogeneous System.*

The subject for consideration is the system of equations (2) which in the notation of §2 becomes

$$\frac{d\xi}{dt}=S\varkappa(t)\xi, \tag{4}$$

where $\varkappa(t)=(k_{ij}(t))_{i,j=1,2,\ldots}$ is a matrix each of whose elements is a power series in the complex variable t converging for $|t|\leqq r$. Throughout the discussion it will be assumed that $\varkappa(t)$ satisfies the hypothesis

(H_1). *The coefficients* $\varkappa_n=(k_{ijn})_{i,j=1,2,\ldots}$ *in the matrix equation*

$$\varkappa(t)=\sum_{n=0}^{\infty}\varkappa_n t^n,$$

are such that $\varkappa_n$ *is limited* l_n, *and moreover,*

$$\sum_{n=0}^{\infty} l_n r^n$$

converges. Therefore there exists a number N *such that* $l_n\leqq Nr^n$.

PROPOSITION 1. *The matrix* $\varkappa(t)$ *is limited for* $|t|\leqq r$ *and its limit may be taken as the quantity* $l(t)$ *defined by*

$$l(t)=\sum_{n=0}^{\infty} l_n |t|^n.$$

To establish this result, recall the definition of a limited matrix. Let ξ and η be two points of Hilbert space. Then it follows that

$$\begin{aligned}|S_mS_m\xi\varkappa(t)\eta| &= |S_mS_m\xi(\sum_{n=0}^{\infty}\varkappa_n t^n)\eta| = |\sum_{n=0}^{\infty} t^n(S_mS_m\xi\varkappa_n\eta|,\\ &\leqq M\xi M\eta(\sum_{n=0}^{\infty} l_n|t|^n)=l(t)\,(M\xi M\eta).\end{aligned} \tag{5}$$

It is seen that (5) states the limited character of $\varkappa(t)$ and that $l(t)$ may be taken as the limit.

PROPOSITION 2. *For every* i *the series*

$$\sum_{g=1}^{\infty}\bar{k}_{ig}(t)k_{ig}(t) \tag{6}$$

converges uniformly for all $|t|\leqq r$. *Similarly, for every* g *the same condition is satisfied by the series*

$$\sum_{i=1}^{\infty}\bar{k}_{ig}(t)k_{ig}(t). \tag{7}$$

From the fact that $k_{ig}(t)=\Sigma_{n=0}^{\infty}k_{ign}t^n$ it is seen that

$$\sum_{g=m_1}^{m_2}\bar{k}_{ig}(t)k_{ig}(t)=\sum_{n,h=0}^{\infty}\bar{t}^n t^h(\sum_{g=m_1}^{m_2}\bar{k}_{ign}k_{igh})\ll\sum_{n,h=0}^{\infty}l_hl_n\bar{t}^nt^h, \tag{8}$$

where the dominance relation is a consequence of Property (4) in view of the assumption of (H_1) as to the matrices x_n. As a result of (8) it may be stated that for every assigned number $l>0$ an integer n_l can be determined so that for all m_1 and m_2 and for $|t|\leqq r$,

$$|\sum_{n,\,h=n_l}^{\infty} \bar{t}^n t^h (\sum_{g=m_1}^{m_2} \bar{k}_{ign} k_{igh})| \leqq \frac{l}{2}. \tag{9}$$

Because of the convergence for all n and h of the series $\Sigma_{g=1}^{\infty} \bar{k}_{ign} k_{igh}$, an integer m_l can be chosen so large that, for $m_1 \geqq m_l$, $m_2 \geqq m_l$, and for $|t|\leqq r$,

$$|\sum_{n,\,h=0}^{n_l} \bar{t}^n t^h (\sum_{g=m_1}^{m_2} \bar{k}_{ign} k_{igh})| \leqq \frac{l}{2}. \tag{10}$$

From (9) and (10) it follows that the series in (8) is in absolute value at most l for $|t|\leqq r$, which establishes the proposition.

PROPOSITION 3. *Let t_0 be a point in the region $|t|\leqq r$. If $\lambda(t-t_0)$ represents the matrix obtained from $x(t)$ by expanding each element of $x(t)$ as a series in $(t-t_0)$, then the coefficient matrices λ_n in*

$$\lambda(s)=\sum_{n=0}^{\infty}\lambda_n s^n, \qquad (s=t-t_0),$$

are limited by numbers l'_n, and $\Sigma_{n=0}^{\infty} l'_n r_0^n$ converges, where r_0 is any value satisfying $r_0<r-|t_0|$.

In other words, Proposition (3) states that the analytic continuation of $x(t)$ to the point t_0 satisfies, for $|s|\leqq r_0$, conditions which are precisely similar to those satisfied by $x(t)$ for $|t|\leqq r$.

Let the notation for the analytic continuations of the elements of $x(t)$ be

$$k_{ij}(t)=\sum_{n=0}^{\infty} h_{ijn} s^n,$$

so that $\lambda(s)=\Sigma_{n=0}^{\infty}\lambda_n s^n$, where $\lambda_n=(h_{ijn})_{i,j=1,2,\ldots}$. In order to show that the λ_n are limited and that the limits satisfy the convergence condition of the proposition, recall Definition 2. Let ξ and η be two points of Hilbert space. Then

$$S_m S_m \xi x(t)\eta=\sum_{n=0}^{\infty} C_n t^n, \qquad (C_n=S_m S_m \xi x_n \eta). \tag{11}$$

From (H_1) it follows that

$$|C_n|\leqq l_n M\xi M\eta,$$

and, hence,

$$\sum_{n=0}^{\infty} C_n t^n \ll (M\xi M\eta)\sum_{n=0}^{\infty} l_n t^n, \tag{12}$$

where the right side converges for $|t|\leqq r$. Consequently, if l'_n represents the

coefficient of s^n in the series formed by continuing the series on the right side of (12) to the point $t=|t_0|$, the analytic continuation of the series on the left to the point $t=t_0$ satisfies

$$\sum_{n=0}^{\infty} C_n t^n = \sum_{n=0}^{\infty} B_n s^n \ll \Big(\sum_{n=0}^{\infty} l'_n s^n\Big) M\xi M\eta, \qquad (|s|<r-|t_0|). \tag{13}$$

But, $B_n = S_m S_m \xi \lambda_n \eta$, and, hence, it has been established that

$$|S_m S_m \xi \lambda_n \eta| \leqq l'_n M\xi M\eta.$$

From (13) it is seen that $\sum_{n=0}^{\infty} l'_n r_0^n$ converges if $r_0<r-|t_0|$.

As a consequence of Proposition (3) the theorems stated below, where the hypotheses refer to the initial point $t=0$, apply to the case $t=t_0$, if r is changed to r_0 and l_n to l'_n.

[illegible]. *The Existence Theorem for the System* (4).

The fundamental theorem concerning the solution of (4) involves functions $\xi(t)$ of a type which it will be convenient to represent by the class $\mathfrak{M}$.

DEFINITION (4). *A function $\xi(t)$ belongs to the class $\mathfrak{M}$ in case there exists a $w(0<w\leqq r)$ so that, for $|t|\leqq w$, $\xi(t)$ belongs to Hilbert space, and if the coordinates $x_i(t)$ of $\xi(t)$ are convergent power series, $x_i(t)=\sum_{n=0}^{\infty} x_{in}t^n$, or*

$$\xi(t)=\sum_{n=0}^{\infty} \xi_n t^n, \qquad [\xi_n=(x_{1n}, x_{2n}, \ldots .)]. \tag{14}$$

Moreover, it is possible to interchange infinite summations in the following expression and to write

$$S\varkappa(t)\xi(t) = S\Big(\sum_{n=0}^{\infty} \varkappa_n t^n\Big)\Big(\sum_{m=0}^{\infty} \xi_m t^m\Big) = \sum_{n,\,m}^{0,\,\infty} t^{n+m} S\varkappa_n \xi_m . \tag{15}$$

A function $\xi(t)$ will be said to belong to the class $\mathfrak{M}$ at t_1 if it satisfies conditions corresponding to those just enumerated with $\varkappa(t)$ replaced by its analytic continuation to the point t_1.

PROPOSITION 4. *Suppose that $\xi(t)$ is a function satisfying* (14) *and that, instead of* (15), *there holds*

$$M\xi(t)\leqq K<\infty, \qquad (|t|\leqq w). \tag{16}$$

Then it follows that $\xi(t)$ satisfies the condition (15).

Let the proposition be established with the aid of Property (2). Identify the $\xi(u)$ and $\eta(u)$ of that result with $\xi(t)$ and the point $(k_{i1}(t), k_{i2}(t), \ldots .)$, respectively. In view of Proposition (2) and since $M\xi(t)\leqq K$, it follows that, as a consequence of Property (2), the series

$$\sum_{j=1}^{\infty} k_{ij}(t)x_j(t) = \sum_{j=1}^{\infty} \Big(\sum_{n=0}^{\infty} k_{ijn}t^n\Big)\Big(\sum_{m=0}^{\infty} x_{jm}t^m\Big)$$

converges uniformly. Therefore it may be rearranged as a power series in t; in other words $\xi(t)$ satisfies (15).

The existence of a solution of (4) taking on a given set of values for $t=0$ is established in

THEOREM I. *Let the system* (4) *satisfy* (H_1) *and suppose that* ξ_0 *is a point in Hilbert space. Then, among all functions of the class* $\mathfrak{M}$ *there exists uniquely for* $|t|<r$, *a function* $\xi(t)$ *which is a solution of* (4) *satisfying* $\xi(0)=\xi_0$. *Furthermore, the coefficients* ξ_n *in the expansion of* $\xi(t)$,

$$\xi(t)=\sum_{n=0}^{\infty}\xi_n t^n,$$

are given by $\xi_n=S\lambda_n\xi_0$ *where the matrices* λ_n *are the coefficients in the expansion of a certain matrix* $\lambda(t)=\Sigma_{n=0}^{\infty}\lambda_n t^n$ *satisfying the following conditions for* $|t|<r$:

The matrices λ_n *are limited* b_n *and the series*

$$l'(t)=\sum_{n=0}^{\infty}b_n|t|^n \tag{17}$$

converges for $|t|<r$.

Moreover, $\xi(t)$ *can be written in the form*

$$\xi(t)=S\lambda(t)\xi_0,$$

and the series giving the respective components $x_i(t)$ *of* $\xi(t)$ *converge in* t, *uniformly with respect to the index* i.

The theorem will be established by the method of undetermined coefficients. Suppose there exists a solution $\xi(t)$ of (4) belonging to the class $\mathfrak{M}$ and satisfying $\xi(0)=\xi_0$. Then it follows that

$$\xi(t)=\sum_{n=0}^{\infty}\xi_n t^n,\quad \frac{d\xi(t)}{dt}=\sum_{n=1}^{\infty}n\xi_n t^{n-1}.$$

Hence there is obtained

$$\begin{aligned}\sum_{n=1}^{\infty}n\xi_n t^{n-1}&=Sx(t)\Big(\sum_{n=0}^{\infty}\xi_n t^n\Big)=S\Big(\sum_{n=0}^{\infty}x_n t^n\Big)\Big(\sum_{m=0}^{\infty}\xi_m t^m\Big)\\&=\sum_{n,\,m=0}^{\infty}(Sx_n\xi_m)t^{n+m}=\sum_{g=0}^{\infty}C_g t^g,\qquad \Big(C_g=\sum_{\substack{n,\,m=0\\ n+m=g}}^{g}Sx_n\xi_m\Big),\end{aligned}\tag{18}$$

where the interchange of S and the infinite summations with respect to m and n is seen to be legitimate because $\xi(t)$ belongs to the class $\mathfrak{M}$. The equation (18) serves to determine uniquely the coefficients ξ_n; there results

$$\xi_0=\xi_0,\quad n\xi_n=C_{n-1},\qquad (n=1,2,\ldots.), \tag{19}$$

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which gives

$$\left.\begin{array}{ll}\xi_1=S\varkappa_0\xi_0=S\lambda_1\xi_0\,, & (\lambda_1=\varkappa_0)\,,\\ 2\xi_2=S\varkappa_1\xi_0+S^2\varkappa_0\lambda_1\xi_0=2S\lambda_2\xi_0\,, & \left(\lambda_2=\dfrac{\varkappa_1+S\varkappa_0\lambda_1}{2}\right),\\ \dots\dots\dots\dots\dots\dots\dots\dots, & \dots\dots\dots\dots\dots,\\ n\xi_n=S\varkappa_{n-1}\xi_0+\dots\dots & \\ \qquad\qquad+S\varkappa_0\xi_{n-1}=nS\lambda_n\xi_0\,, & \left(\lambda_n=\dfrac{\varkappa_{n-1}+S\varkappa_{n-2}\lambda_1+\dots\dots+S\varkappa_0\lambda_{n-1}}{n}\right).\end{array}\right\}\quad(20)$$

In order to prove the theorem it will first be shown that the series giving the components $x_i(t)$ of the function $\xi(t)=\Sigma_{n=0}^{\infty}\xi_n t^n$, resulting from the coefficients ξ_n found in (20), converge absolutely for $|t|<r$ and uniformly with respect to the index i.

Recall that the matrix $\varkappa_n$ is limited $l_n \leqq N/r^n$; in the work below it will be assumed that the limit is equal to N/r^n. Then, because of Definition 2, it follows that

$$|\xi_1|\leqq NM\xi_0=b_1M\xi_0, \qquad (b_1=N).$$

From Property (6) of limited matrices it is seen that λ_2 is limited b_2,

$$b_2=\frac{N}{2}\left(N+\frac{1}{r}\right),$$

and hence $|\xi_2|\leqq b_2M\xi_0$. On proceeding in this manner it can be established that every matrix λ_n is limited b_n, where

$$b_n\ =\frac{1}{n}\left(\frac{N}{r^{n-1}}+b_1\frac{N}{r^{n-2}}+\dots\dots+b_{n-1}N\right),$$

$$b_{n+1}=\frac{1}{n+1}\left(\frac{N}{r^n}+b_1\frac{N}{r^{n-1}}+\dots\dots+b_{n-1}\frac{N}{r}+b_nN\right).$$

It follows that there is the relation

$$b_{n+1}=\frac{b_n}{n+1}\left(\frac{n}{r}+N\right),$$

and that $|\xi_n|\leqq b_nM\xi_0$.

LEMMA 1. *The series $\Sigma_{n=0}^{\infty}b_nt^n$ converges for $|t|<r$.*

To see this consider the absolute value q_n of the ratio of the *n-th* to the $(n-1)$*-th* term:

$$q_n=\frac{b_n}{b_{n-1}}|t|=\left(\frac{n-1}{n}\right)\left(\frac{N_r}{n-1}+1\right)\left|\frac{t}{r}\right|.$$

Since for $|t|<r$ $\lim_{n=\infty}q_n<1$, the lemma is established. Moreover, since

$$\xi(t)=\sum_{n=0}^{\infty}\xi_nt^n=\sum_{n=0}^{\infty}(S\lambda_n\xi_0)t^n\ll\xi_0+M\xi_0\left(\sum_{n=1}^{\infty}b_nt^n\right), \qquad (\lambda_0=1), \qquad (21)$$

it follows that the series for each component $x_i(t)$ of $\xi(t)$ converges just as fast and in the same region as the series in Lemma 1.

Now define a matrix $\lambda(t)$ formally by the equation

$$\lambda(t)=\sum_{n=0}^{\infty}\lambda_n t^n, \qquad \{\lambda(t)=(m_{ij}(t))_{i,j=1,2,\ldots}\}. \tag{22}$$

The series in (22) (one for each component of $\lambda(t)$) converge, because of Lemma (1), since $|\lambda_n|\leqq b_n$. The matrix $\lambda(t)$ therefore satisfies conditions corresponding to those imposed on $\kappa(t)$ in hypothesis (H_1). Hence $\lambda(t)$ is limited $l'(t)$ where $l'(t)$ has the definition (17). From the fact that $\lambda(t)$ is limited uniformly for $|t|\leqq r_1<r$, it follows that $S\lambda(t)\xi_0$ converges uniformly and, therefore,

$$S\lambda(t)\xi_0=S(\sum_{n=0}^{\infty}\lambda_n t^n)\xi_0=\sum_{n=0}^{\infty}(S\lambda_n\xi_0)t^n=\xi(t).$$

To complete the proof of the theorem it remains to show that $\xi(t)$ satisfies (15). First consider

COROLLARY (1). *The modulus of the function $\xi(t)$ satisfies the equation*

$$M\xi(t)=\sqrt{S\bar{\xi}(t)\xi(t)}\leqq l'(t)M\xi_0.$$

It has been seen that $\xi(t)=S\lambda(t)\xi_0$. Hence, if α is a point in Hilbert space, it follows from Definition (2) that, since $\lambda(t)$ is limited $l'(t)$,

$$|S\alpha\xi(t)|=|S\alpha(S\lambda(t)\xi_0)|\leqq M\alpha(l'(t)M\xi_0).$$

Hence, in view of Property (1) it follows that $M\xi(t)$ has the upper bound stated in the corollary.

It is now easily verified that $\xi(t)$ satisfies the hypothesis of Proposition (4) and therefore belongs to the class $\mathfrak{M}$. This completes the proof of the theorem.

COROLLARY (2). *The solution $\xi(t)$ has the property that the two series*

$$M\xi(t)=\sum_{i=1}^{\infty}\bar{x}_i(t)x_i(t), \quad M\left(\frac{d\xi(t)}{dt}\right)=\sum_{i=1}^{\infty}\overline{\frac{dx_i(t)}{dt}}\,\frac{dx_i(t)}{dt}, \tag{23}$$

converge uniformly for $|t|\leqq r_1$ for all $r_1<r$.

There is obtained

$$|\sum_{i=n}^{m}\bar{x}_i(t)x_i(t)|=|\sum_{i=n}^{m}(\sum_{h,j=0}^{\infty}\bar{x}_{ij}\bar{t}^j x_{ih}t^h)|$$

$$\leqq\sum_{h,j=0}^{\infty}|t|^{h+j}|\sum_{i=n}^{m}\bar{x}_{ij}x_{ih}|\ll(M\xi_0)^2(\sum_{h,j=0}^{\infty}b_h b_j|t|^{h+j}),$$

where the dominance relation is a consequence of the fact that $\xi_n=S\lambda_n\xi_0$. Then, to prove the uniform convergence of $S\bar{\xi}(t)\xi(t)$ one would proceed from this point on precisely as in the work leading to (9) and (10) of Proposition (2).

The proof of the uniform convergence of the other series of the corollary is of similar nature.

The corollary to Theorem I and the Theorem II which follow are of importance in the consideration of fundamental sets of solutions.

COROLLARY (3). *Let $\theta(t)$ be a matrix $(x_{ij}(t))$ in which every column constitutes a solution of* (4). *Then, if all these solutions belong to the class $\mathfrak{M}$, the matrix satisfies the equation*

$$\theta(t) = S\lambda(t)\theta(0). \tag{24}$$

The equation (24) is an obvious result of the condition which holds for every column of $\theta(t)$.

THEOREM II. *Let $\theta(t)$ be a matrix in which each column constitutes a solution of* (4) *belonging to the class $\mathfrak{M}$. Suppose that $\theta(0)$ is limited by the quantity $L(0)$. Then $\theta(t)$ is limited $L(t)$,*

$$L(t) = L(0)l'(t). \tag{25}$$

To establish this result refer to Corollary (3) to Theorem I. It is seen from Property (6) of § 2 that, as a consequence of equation (24), $\theta(t)$ is limited by the product of the limits of $\lambda(t)$ and $\theta(0)$, which is the result stated in the theorem.

§ 4. *The Adjoint System.*

In the consideration of a finite system of linear differential equations the solutions of the adjoint system are found to have certain interesting properties in relation to the solution of the given system. On carrying over to infinite systems the classical definition of adjoint systems, there is obtained

DEFINITION 5. *The system of linear differential equations adjoint to the system* (4) *is the system*

$$\frac{d\xi}{dt} = -Sx'(t)\xi, \tag{26}$$

where $x'(t)$ is the transposed matrix of $x(t)$.

The hypothesis (H_1) was symmetrical in its conditions on the rows and columns of $x(t)$ so that, as a consequence, system (26) is of the same type as (4). A useful relation between the solutions of (4) and (26) is

THEOREM III. *Let $\xi_1(t)$ and $\xi_2(t)$ be two functions belonging to the class $\mathfrak{M}$ for $|t| \leqq r_1 < r$ which are, respectively, solutions of the equations* (4) *and* (26). *Then*

$$\frac{d}{dt}\{S\xi_1(t)\xi_2(t)\} = 0.$$

On formally differentiating $S\xi_1\xi_2$, there is obtained

$$\frac{d}{dt}\{S\xi_1(t)\xi_2(t)\}=S\left(\frac{d\xi_1}{dt}\right)\xi_2+S\xi_1\left(\frac{d\xi_2}{dt}\right). \tag{27}$$

Assume for the present that the term by term differentiation is legitimate. Then, on substituting the values of $d\xi_1/dt$ and $d\xi_2/dt$, it follows that

$$\begin{aligned}\frac{d}{dt}\{S\xi_1(t)\xi_2(t)\} &= S\{S\varkappa(t)\xi_1(t)\}\xi_2(t)-S\xi_1(t)\{S\varkappa'(t)\xi_2(t)\}\\ &= S^2\xi_2(t)\varkappa(t)\xi_1(t)-S\xi_1(t)\{S\xi_2(t)\varkappa(t)\}\\ &= S^2\xi_2(t)\varkappa(t)\xi_1(t)-S^2\xi_2(t)\varkappa(t)\xi_1(t)=0,\end{aligned}$$

where the changes in the forms of expressions are justified by Property (5) of § 2.

To justify the term by term differentiation in (27), recall Property (2) and consider proving that the infinite series on the right in (27) converge uniformly. For the first term, identify the $\eta(u)$ and $\xi(u)$ of Property (2) with $\xi_2(t)$ and $d\xi_1/dt$, respectively. The hypothesis concerning η is satisfied because of Corollary (2) to Theorem I. Since

$$\frac{d\xi_1(t)}{dt}=S\varkappa(t)\xi_1(t)=S\varkappa(t)\{S\lambda(t)\xi_0\}, \qquad (\xi_1(0)=\xi_0),$$

it follows that the modulus of $d\xi_1/dt$ is at most $l(t)l'(t)M\xi_0$ which is uniformly bounded for $|t|\leqq r_1$. Hence the first term on the right in (27) converges uniformly and, similarly, it is established that the second term has the same property. Consequently, the term by term differentiation leading to (27) was legitimate and the proof of the theorem is complete.

§ 5. *Fundamental Sets of Solutions.*

In considering finite systems of linear differential equations the notion of fundamental sets of solutions is of great importance. In the definition of such sets, the determinant of the matrix formed by the set plays a primary rôle. On replacing the idea of the non-vanishing of this determinant by the notion of a limited matrix possessing a unique limited reciprocal matrix there is immediately suggested, as an analogue for (4) of the classical definition of fundamental sets,

DEFINITION 6. *A matrix of solutions $\theta(t)$ of* (4) *will be termed a fundamental set of solutions at $t=t_0$ in case*

$\theta(t_0)$ is a limited matrix, and in case (28)

there exists a unique limited reciprocal matrix for $\theta(t_0)$. (29)

In establishing the fact that a set of solutions, fundamental at t_0, is fundamental for all $|t|<r$, there will be needed

THEOREM IV. *Let $\xi(t)$ be the solution of* (4) *defined by the initial condition*

$$\xi(t_0)=\xi_0, \quad (|t_0|<r). \tag{30}$$

Then the analytic continuation of $\xi(t)$ to the point $t=0$ can be obtained, and the expansion there converges for all $|t|<r$.

This result is obtained by recalling Proposition (3). The condition (30) defines a solution in the neighborhood of t_0, and this solution can then be continued to the point $t=0$ by successive applications of Proposition (3) and Theorem I.

THEOREM V. *If $\theta(t)$ is a fundamental set at $t=0$, and if ξ_0 is a point in Hilbert space, then there exists a point η of Hilbert space such that the solution $\xi(t)$ of* (4) *defined by the initial condition $\xi(0)=\xi_0$ is given by $\xi(t)=S\theta(t)\eta$.*

Let η be determined so that $S\theta(0)\eta=\xi_0$, where, since $\theta(0)$ has a unique reciprocal $\phi(0)$, it follows from Property (9) that $\eta=S\phi(0)\xi_0$. Consequently, there is obtained

$$\xi(t)=S\lambda(t)\xi_0=S\lambda(t)\,(S\theta(0)\eta)=S(S\lambda(t)\theta(0))\eta=S\theta(t)\eta,$$

where the interchange of summations is effected by means of Property (5).

The definition given for fundamental sets of solutions is seen to have content because the elements of a matrix of solutions $\theta(t)=(x_{ij}(t))$ can be defined by the initial conditions

$$x_{ij}(0)=d_{ij}, \quad (d_{ii}=1;\ d_{ij}=0,\ i\neq j). \tag{31}$$

For $t=0$, $\theta(t)$ reduces to the unit matrix which is limited and possesses a unique limited reciprocal matrix. As a matter of fact, the matrix $\lambda(t)$ of Theorem I can easily be shown to be the matrix defined by the conditions (31).

With the aid of Theorem III consider the proof of the following theorem which states an important property of fundamental sets of solutions of (4).

THEOREM IV. *Let $\theta(t)$ be a matrix of solutions of* (4) *which is fundamental at $t=t_0$. Then, the analytic continuation of $\theta(t)$ to the point t_1 is fundamental at t_1 for all $|t_1|<r$.*

In the first place, it is easily seen that $\theta(t_1)$ is limited for all $|t_1|<r$, because $\theta(t)$ can be continued to the point $t=0$, and remains limited at every point reached in the continuation. Theorem II then shows that $\theta(t)$ is limited $L(0)l'(t)$ where $L(0)$ is the limit of the matrix $\theta(0)$. It remains to prove that $\theta(t_1)$ possesses a unique reciprocal matrix.

Let $\theta_{(t_0)}^{-1}$ be the reciprocal matrix of $\theta(t_0)$. Define a matrix $\phi(t)$ of solutions of the adjoint system (26) by the initial condition $\phi'(t_0)=\theta_{(t_0)}^{-1}$, where ϕ' represents the transposed matrix of ϕ. As a consequence of Theorem III it follows that in a sufficiently small neighborhood of t_0 the ϕ and θ satisfy

$$\frac{d}{dt}(S\phi'(t)\theta(t))=0.$$

Hence, in this neighborhood the product matrix $S\phi'(t)\theta(t)$ must remain constant and, therefore, is always equal to the unit matrix since this is the value it assumes for $t=t_0$. Since $\phi'(t)$ is limited, $\theta(t)$ therefore possesses a limited left-hand reciprocal $\theta_{(t)}^{-1}=\phi'(t)$ for all t in the restricted neighborhood of t_0. By analytic continuation of the matrices ϕ and θ this property is seen to hold for all $|t|<r$. It is easily seen that the existence of two distinct left-hand reciprocals of $\theta(t)$ for any value of t would imply a similar condition at $t=t_0$. Hence there exists a unique left-hand reciprocal $\theta^{-1}(t)=\phi'(t)$ for all $|t|<r$. By Property (8) of § 2 it follows that ϕ' is also the unique right-hand reciprocal of $\theta(t)$ for $|t|<r$.

§ 6. *The Non-Homogeneous System.*

Consider the infinite system

$$\frac{d\xi}{dt}=S\varkappa(t)\xi+\gamma(t), \tag{32}$$

where the function $\gamma(t)$ satisfies the conditions specified in hypothesis

(H_2). *The components $y_i(t)$ of $\gamma(t)$ are analytic in t and are regular for $|t|\leqq r$. Furthermore, the coefficients γ_n in the expansion for $\gamma(t)$,*

$$\gamma(t)=\sum_{n=0}^{\infty}\gamma_n t^n,$$

are points in Hilbert space for which $M\gamma_n\leqq b_n$ where $\Sigma_{n=0}^{\infty}b_n r^n$ exists.

It is easily verified that the solutions of the systems (32) and (4) are related in the fashion stated in

THEOREM V. *Let $\xi_1(t)$ and $\xi_2(t)$ be two solutions of* (32) *belonging to Hilbert space for a certain range of values $|t|\leqq r_1<r$. Then $\xi_3=\xi_2-\xi_1$ is a solution of the system* (4).

PROPOSITION 5. *The function $\gamma(t)$ belongs to Hilbert space for every value of $|t|\leqq r$, and $M\gamma(t)\leqq\Sigma_{n=0}^{\infty}b_n|t|^n$.*

To show this, recall Property (1); it is seen that, if α is a point in Hilbert space, then

$$|S_m\alpha\gamma(t)|=|\sum_{n=0}^{\infty}(S_m\alpha\gamma_n)t^n|\leqq M\alpha(\sum_{n=0}^{\infty}b_n|t|^n). \tag{33}$$

From (33) it follows that $\gamma(t)$ is a Hilbert point for $|t| \leqq r$ and that the modulus $M\gamma(t)$ satisfies the inequality of the proposition.

PROPOSITION 6. *The series*

$$M\gamma(t) = \sum_{i=1}^{\infty} \bar{y}_i(t) y_i(t) \tag{34}$$

converges uniformly for $|t| \leqq r$.

It is verified that

$$\sum_{i=h}^{k} \bar{y}_i(t) y_i(t) \leqq \sum_{n,\, m=0}^{\infty} |t|^{n+m} \left| \sum_{i=h}^{k} \bar{y}_{in} y_{im} \right|, \tag{35}$$

where $\gamma_n = (y_{1n}, y_{2n}, \ldots.)$. As a consequence of hypothesis (H_2) it is seen that the series on the right in (35) is dominated by the series

$$\sum_{n,\, m=0}^{\infty} r^{n+m} (b_n b_m).$$

A discussion similar to that used in the consideration of Proposition 2, § 3, then establishes the uniform convergence of (34).

PROPOSITION 7. *The series giving the components* $y_i(t)$ *of* $\gamma(t)$ *converge in t uniformly with respect to the index i.*

This result follows from the fact that the series for $y_i(t)$,

$$y_i(t) = \sum_{n=0}^{\infty} y_{in} t^n,$$

is dominated by $\Sigma_{n=0}^{\infty} b_n t^n$.

PROPOSITION 8. *Let the notation for the analytic continuation of* $\gamma(t)$ *to the point* t_0 *be*

$$\gamma(t) = \alpha(s) = \sum_{n=0}^{\infty} \alpha_n s^n, \qquad (s = t - t_0;\ |s| < r - |t_0|).$$

Then it follows that the α_n *are points in Hilbert space with moduli* $M\alpha_n \leqq a_n$, *where* $\Sigma_{n=0}^{\infty} a_n |s|^n$ *converges for all* $|s| < r - |t_0|$.

The proposition states that the analytic continuation of $\gamma(t)$ satisfies conditions similar to those imposed on γ in (H_2). The proof will not be given since it would be identical in method with that given for Proposition (3).

The existence theorem for system (32) will be established by a method analogous to the classical method of *variation of parameters* used for a finite system of differential equations. In the proof use will be made of the results obtained in the preceding articles with reference to fundamental sets of solutions and concerning the adjoint system. In view of Theorem V it is necessary to exhibit only one particular solution of (32). The existence of a solution taking on the initial value $\xi(0) = 0$ is established in

THEOREM VI. *Let hypotheses* (H_1) *and* (H_2) *be satisfied. Then there exists uniquely a function* $\xi(t)$ *of the class* $\mathfrak{M}$, *for* $|t|<r$, *satisfying the equation* (32) *and assuming the initial value* $\xi(0)=0$.

Let $\theta(t)$ be a fundamental set of solutions of the homogeneous system (4), and let $\phi(t)$ be the reciprocal of $\theta(t)$. Recall the analytic character of the matrix ϕ which follows from the fact that it constitutes a matrix of solutions of the adjoint system (26). Consider the infinite system of linear equations

$$\xi=S\theta(t)\eta \tag{36}$$

as giving a transformation of the variable ξ in (32) from ξ to η. Then

$$\frac{d\xi}{dt}=S\frac{d\theta(t)}{dt}\eta+S\theta(t)\frac{d\eta}{dt}, \tag{37}$$

provided that the term by term differentiation was legitimate. On substituting (36) and (37) in (32) there is obtained

$$S\frac{d\theta}{dt}\eta+S\theta\frac{d\eta}{dt}=S\varkappa(t)\,(S\theta(t)\eta)+\gamma(t)=S\{S\varkappa(t)\theta(t)\}\eta+\gamma(t), \tag{38}$$

where the interchange on the right is justified by Property (5) of § 2. Since

$$\frac{d\theta(t)}{dt}=S\varkappa(t)\theta(t),$$

equation (38) reduces to

$$S\theta(t)\frac{d\eta}{dt}=\gamma(t). \tag{39}$$

The equation (39) becomes, in view of Property (9) of § 2,

$$\frac{d\eta}{dt}=S\phi(t)\gamma(t). \tag{40}$$

The equation (40) has the solution

$$\eta(t)=\int_0^t S\phi(t)\gamma(t)\,dt, \tag{41}$$

where the integration may be taken along the straight line in the t-plane leading from $t=0$ to $t=t$. In case it can be established that the η of (41) belongs to Hilbert space for $|t|<r$, and is of such a nature that the term by term differentiation in (37) was legitimate, it then follows that $\xi(t)=S\theta(t)\eta(t)$ is a solution of (32). To complete the proof it would remain to show that $\eta(t)$ is a function of the class $\mathfrak{M}$.

Let Property (2) of § 2 be applied to $S\phi(t)\gamma(t)$. Every row of $\phi(t)$ is a solution of the adjoint system (26) and hence, in view of Corollary (1) to Theorem I as applied to (26), the modulus of each row is uniformly bounded for $|t|\leqq r_1<r$. Therefore, since $S\bar{\gamma}(t)\gamma(t)$ converges uniformly, it follows that $S\phi(t)\gamma(t)$ converges uniformly, and hence represents an analytic function.

Consequently, every coordinate of $\eta(t)$ is an analytic function regular for $|t|<r$. Moreover, if α is a point in Hilbert space,

$$|S_m\alpha\eta(t)|\leqq\left|\int_0^t S_m\alpha\{S\phi(t)\gamma(t)\}dt\right|\leqq M\alpha\left|\int_0^t M\gamma(t)L(t)dt\right|,$$

where $L(t)$ is the limit of the matrix $\phi(t)$. As a result of Property (1) of § 2 it follows that $\eta(t)$ is a Hilbert point for $|t|<r$, and its modulus is uniformly bounded for $|t|\leqq r_1(r_1<r)$ in view of the similar property which holds for $M\gamma(t)$ and $L(t)$.

To justify the equation (37) for the $\eta(t)$ of (41) it must be shown that the two series on the right in (37) converge uniformly. From (40) it is seen that the modulus of $d\eta/dt$ is at most $M\gamma(t)L(t)$ which is uniformly bounded for $|t|\leqq r_1<r$. As a result of Corollary (2) to Theorem I, the two infinite series

$$S\left(\overline{\frac{d\theta'(t)}{dt}}\right)\left(\frac{d\theta(t)}{dt}\right),\ S\bar{\theta}'(t)\theta(t),$$

converge uniformly since every column of θ is a solution of (4). Hence it is seen that Property (2) of § 2 may be applied to each of the series in (37) to establish their uniform convergence.

On substituting (41) in (36) it follows that

$$\xi(t)=S\theta(t)\eta(t)=S\theta(t)\left[\textstyle\int_0^t S\phi(t)\gamma(t)dt\right] \tag{42}$$

is a solution of (32) which satisfies $\xi(0)=0$. Since $M\eta(t)$ was uniformly bounded for $|t|\leqq r_1<r$ it follows that $M\xi(t)$ will also be uniformly bounded. It then is seen that, as a consequence of Proposition (4), the function $\xi(t)$ belongs to the class $\mathfrak{M}$.

The solution given by (42) is the only solution of (32) for which $\xi(0)=0$ because, if there were another solution $\xi_1\neq\xi$, it would follow that the difference $\xi_2=\xi-\xi_1$ would be a solution of (4) for which $\xi_2(0)=0$, and therefore $\xi_2(t)=0$ for $|t|<r$.

The existence of a solution of (32) could also be established directly by the method of undetermined coefficients, but the proof of the result is much more difficult than the corresponding proof for the system (4).

Various problems are suggested in connection with systems (4) and (32) by the many known properties of finite systems of linear differential equations. Moreover, systems in which the variables are real and the coefficients are real and continuous may be considered under conditions analogous to those of (H_1) and (H_2).

HARVARD UNIVERSITY, *February* 1, 1917.

*On the Asymptotic Value of Sums of Power Residues.**

BY HOWARD H. MITCHELL.

1. In this paper certain limits will be obtained for the sum of all the positive integers less than a given prime m, the indices of which, with respect to a primitive root of m have the same residue, mod 2ν, where 2ν is a divisor of $m-1$. If $(m-1)/2\nu$ is even, ind $(m-r) \equiv$ ind r, mod 2ν, and hence the value of a sum of the type mentioned is $m(m-1)/4\nu$. We shall therefore suppose that $(m-1)/2\nu$ is odd.

By means of the limits obtained for these sums it will be found that if S_i denotes any one of them, and if ν remains fixed while m ranges over the primes that are congruent to $2\nu+1$, mod 4ν,† then

$$\lim_{m\doteq\infty}\left[\frac{S_i}{\dfrac{m(m-1)}{4\nu}}\right]=1.$$

Perhaps the most interesting result that has been established concerning the sums of the type considered is that if m be a prime of the form $4n+3$ the sum of the quadratic non-residues of m that lie between 0 and m always exceeds the sum of the residues. From this it follows that the number of the residues between 0 and $m/2$ exceeds the number of the non-residues. These results were established by Dirichlet in connection with his investigation of the class number of quadratic forms.‡

Stern has shown that if m is a prime of the form $8n+3$, the sum of the quadratic non-residues between 0 and m is less than twice the sum of the residues, whereas if it has the form $8n+7$, the sum of the non-residues is less than three times the sum of the residues.§ The latter result, however, is

* Presented to the American Mathematical Society, December 28, 1916.

† Dirichlet has shown that in any arithmetical progression in which the first term and common difference have no common factor there is an infinity of primes (Dirichlet-Dedekind, *Zahlentheorie*, 4th edition, §§ 132–137).

‡ *Ibid.*, § 104.

§ *Journal für Mathematik*, Vol. LXXI (1870), pp. 152, 153; Bachman, *Encyklopädie der Mathematischen Wissenschaften*, Bd. I, p. 569.

trivial, since in whatever way the integers from 1 to $m-1$ be divided into two sets containing $(m-1)/2$ numbers each, the sum of either set is less than three times the sum of the other.

The distribution of the quadratic residues is a problem that is closely related to the values of these sums, and one that has interested a number of writers.*

2. The author's results concerning the sums of the sort described are obtained by use of a certain type of Dirichlet's series. These series are special cases of those that were used in his proof of the infinity of primes in an arithmetical progression.† They were also employed by Kummer in the derivation of the expression for the class number of a cyclotomic realm.‡

Following the notation of Dirichlet–Dedekind,§ we write

$$L^\circ(\alpha) = \sum_z \frac{\alpha^{\operatorname{ind} z}}{z},$$

where the sum is to be taken over all the positive integers z that are not divisible by a given prime m, α denotes any root of the equation $\alpha^\nu = -1$, where 2ν is a divisor of $m-1$, and ind z denotes the index of z, mod m, with respect to a particular primitive root.

In case $(m-1)/2\nu$ is odd, the sum of this series may be expressed in finite form as follows:||

$$L^\circ(\alpha) = -\frac{\pi i}{m^2}(\alpha, \theta)\phi(\alpha).$$

The symbol (α, θ) denotes the Lagrange resolvent function

$$(\alpha, \theta) = \sum_{r=1}^{m-1} \alpha^{\operatorname{ind} r}\theta^r,$$

where $\theta = e^{2\pi i/m}$. The symbol $\phi(\alpha)$ is defined by

$$\phi(\alpha) = -\sum_r r\alpha^{-\operatorname{ind} r},$$

where r assumes the same values.

* Lebesgue, *Journal de Mathèmatiques*, Vol. VII (1842), p. 137; Götting, *Journal für Mathematik*, Vol. LXX (1869), p. 363; Stern, *loc. cit.*; Osborn, *Messenger of Mathematics*, Vol. XXV (1895), p. 45; Glaisher, *Quarterly Journal of Mathematics*, Vol. XXXIII (1902), pp. 319–328; *ibid.*, Vol. XXXIV (1903), pp. 1, 178; Karpinski, *Journal für Mathematik*, Vol. CXXVII (1904), p. 1; Holden, *Messenger of Mathematics*, Vol. XXXV (1905), pp. 102, 110; *ibid.*, Vol. XXXVI (1906), pp. 75, 126.

† *Loc. cit.;* also Werke, Vol. I, p, 313.

‡ *Journal für Mathematik*, Vol. XL, p. 98.

§ P. 625.

|| Combining equations (58), (65), *loc. cit.*

We proceed to obtain an upper limit for the absolute value of $L^\circ(\alpha)$. In consequence of our assumption that $(m-1)/2\nu$ is odd, the series may be written in the form *

$$L^\circ(\alpha)=\sum_r \alpha^{\operatorname{ind} r}\left[\frac{1}{r}-\frac{1}{m-r}+\frac{1}{m+r}-\frac{1}{2m-r}+\ldots\right],$$

where the sum is to be taken over the positive integers r that are less than $m/2$. The absolute value of $L^\circ(\alpha)$ is thus less than

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{\frac{m-1}{2}}<\log\left(\frac{m+1}{2}\right)+C,$$

where C denotes Euler's constant, $C=.577+$.

The absolute value of (α, θ) is equal to $\sqrt{m}$. Hence the absolute value of $\phi(\alpha)$ is less than

$$\frac{m\sqrt{m}}{\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right].$$

The function $\phi(\alpha)$ may be written in the form

$$\phi(\alpha)=-\sum_{i=0}^{2\nu-1} S_i\alpha^{-i},$$

where S_i denotes the sum of the positive integers r less than m for which ind $r\equiv i$, mod 2ν. Since we are assuming that $\alpha^\nu=-1$, this may in turn be written

$$\phi(\alpha)=\sum_{i=0}^{\nu-1}(S_{i+\nu}-S_i)\alpha^{-i}.$$

From this we obtain

$$\nu(S_{i+\nu}-S_i)=\sum_\alpha \alpha^i\phi(\alpha),$$

where the sum is to be taken over all the roots of the equation $\alpha^\nu=-1$, and where i may have any one of the values $0, 1, 2, \ldots, \nu-1$.

We therefore conclude that

$$|S_{i+\nu}-S_i|<\frac{m\sqrt{m}}{\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right].$$

If a number r appears in the sum S_i, then $m-r$ will appear in the sum $S_{i+\nu}$, and hence

$$S_{i+\nu}+S_i=\frac{m(m-1)}{2\nu}.$$

The values of S_i and $S_{i+\nu}$ are thus included between the limits,

$$\frac{m(m-1)}{4\nu}\pm\frac{m\sqrt{m}}{2\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right].$$

* Cf. Glaisher, *Quarterly Journal of Mathematics*, Vol. XXXIII (1902), pp. 306, 307, 317.

From this it is clear that these sums obey an asymptotic law in that if ν is considered to remain fixed, and m to range over the primes that are congruent to $2\nu+1$, mod 4ν, then

$$\lim_{m=\infty}\left[\frac{S_i}{\frac{m(m-1)}{4\nu}}\right]=1.$$

3. Certain extensions of these results are possible in case m is composite. For example, if m is an odd integer of the form $4n+3$ not divisible by a square factor, and $\left(\frac{z}{m}\right)$ denotes the Jacobi symbol for quadratic residues, we have *

$$\sum_z\left(\frac{z}{m}\right)\frac{1}{z}=-\frac{\pi}{m\sqrt{m}}\sum_r\left(\frac{r}{m}\right)r,$$

where the sum on the left is to be taken over all the positive integers z that are prime to m, and the sum on the right over all the positive integers r that are less than m and prime to m.

From this relation we obtain in the same manner as above

$$-\Sigma\left(\frac{r}{m}\right)r<\frac{m\sqrt{m}}{\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right],$$

from which we conclude that the sum of the positive integers r that are less than and prime to m, and for which $\left(\frac{r}{m}\right)=+1$ is included between the limits

$$\frac{m\phi(m)}{4}-\frac{m\sqrt{m}}{2\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right],\quad\frac{m\phi(m)}{4},$$

and the sum of those for which $\left(\frac{r}{m}\right)=-1$ is included between the limits

$$\frac{m\phi(m)}{4},\quad\frac{m\phi(m)}{4}+\frac{m\sqrt{m}}{2\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right],$$

where, as usual, $\phi(m)$ denotes the number of integers less than and prime to m.

If e is any fixed number less than 1, it may be shown that if m be taken sufficiently large, $\phi(m)>m^e$. From this it follows that

$$\frac{m\phi(m)}{4}$$

is an asymptotic value for either of the two sums.

A similar generalization may be given in the case of higher residues.

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* Dirichlet-Dedekind, § 103.

4. By means of the above result we may obtain an upper limit for the number of classes of quadratic forms of determinant $-m$, where m is an odd integer of the form $4n+3$ not divisible by a square factor. This number is given by

$$h=-\frac{1}{m}\left[2-\left(\frac{2}{m}\right)\right]\Sigma\left(\frac{r}{m}\right)r,^{*}$$

and it therefore satisfies the inequality,

$$h<\left[2-\left(\frac{2}{m}\right)\right]\frac{\sqrt{m}}{\pi}\left[\log\left(\frac{m+1}{2}\right)+C\right].$$

For large values of m this is a much lower limit than those obtained by Holden;† in fact it is evident that if e denotes any fixed number greater than 1/2,

$$\lim_{m\doteq\infty}\left[\frac{h}{m^{e}}\right]=0.$$

Similar results may be obtained for the number of classes of ideals in the quadratic realm $k(\sqrt{-m})$.‡

By means of the upper limit found in § 2 for the absolute value of the function $\phi(\alpha)$ an upper limit may also be obtained for the so-called *first factor* of the class number of the cyclotomic realm determined by *m-th* roots of unity.§ Kummer has stated, apparently without proof, that this first factor obeys an asymptotic law.||

An upper limit may also be obtained for the first factor of the class number of the realm of degree 2ν, determined by the 2ν periods formed from *m-th* roots of unity, where $(m-1)/2\nu$ is odd.¶

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*Dirichlet-Dedekind, § 104.

† *Messenger of Mathematics*, Vol. XXXV (1905), p. 106.

‡ Hilbert, "Die Theorie der algebraischen Zahlkörper," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Vol. IV (1894), p. 320.

§ Kummer, *Journal für Mathematik*, Vol. XL (1850), p. 110; Dirichlet-Dedekind, p. 630.

|| *Journal de Mathématiques*, Vol. XVI (1851), p. 473. Cf. also H. J. S. Smith, "Report on the Theory of Numbers," *Works*, Vol. I, p. 114.

¶ Kummer, *Journal für Mathematik*, Vol. XL (1850), p. 112.

On the Structure of Finite Continuous Groups.

By J. A. Bullard.

Introduction.

The structure of a finite continuous group is determined in part by the "characteristic equation of the group," that is, the characteristic equation of the general infinitesimal transformation of the adjoint group of the given group. It is the purpose of this paper to show how certain properties of a group of linear homogeneous transformations can be obtained at once from the characteristic equation of the general infinitesimal transformation of the group, and thus how the type of structure is in part determined immediately from the infinitesimal transformations of such a group without the determination of the adjoint group. In § 1 I show how to find the structural constants, and thus the roots of the characteristic equation, of the general infinitesimal transformation of the adjoint group of the general linear homogeneous group relative to a given subgroup; in § 2 I find similar results for the special linear homogeneous group relative to a given subgroup. These results lead to certain relations, given in § 3, between the characteristic equation of a group of linear homogeneous transformations and the characteristic equation of the general infinitesimal transformation of the given group. By the aid of these relations I establish theorems I–X by which certain properties of such a group can be at once determined from the general infinitesimal transformation of the group.

Section 1.

Let G denote the general linear homogeneous group in n variables. The symbols of the infinitesimal transformations of G are the n^2 differential operators

$$\varepsilon_{ij}=x_i\frac{\partial}{\partial x_j}, \qquad (i, j=1, 2, \ldots, n),$$

or any n^2 linearly independent linear homogeneous functions of these operators with constant coefficients. Let G_r denote any given subgroup of the general linear homogeneous group with r parameters, the symbols of whose infinitesimal transformations are the r given differential operators

$$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_r, \text{ where } \mathscr{A}_h=\sum_{i,j=1}^{n} a_{hij}\varepsilon_{ij}, \qquad (h=1, 2, \ldots, r).$$

Then
$$\left\| \begin{matrix} a_{111}, a_{112}, \ldots, a_{11n}, a_{121}, \ldots, a_{1n1}, a_{1n2}, \ldots, a_{1nn} \\ a_{211}, a_{212}, \ldots, a_{21n}, a_{221}, \ldots, a_{2n1}, a_{2n2}, \ldots, a_{2nn} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{r11}, a_{r12}, \ldots, a_{r1n}, a_{r21}, \ldots, a_{rn1}, a_{rn2}, \ldots, a_{rnn} \end{matrix} \right\| \neq 0,$$

and
$$(\mathscr{A}_i, \mathscr{A}_j) = \mathscr{A}_i\mathscr{A}_j - \mathscr{A}_j\mathscr{A}_i = \sum_{h=1}^{r} c_{jih}\mathscr{A}_h, \qquad (i, j=1, 2, \ldots, r).$$

We may take as the infinitesimal transformations of G the differential operators $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_r$, together with any n^2-r other differential operators $\mathscr{A}_{r+1}, \mathscr{A}_{r+2}, \ldots, \mathscr{A}_{n^2}$ defined as follows:

$$\mathscr{A}_{r+h} = \sum_{i,j=1}^{n} a_{r+h,i,j}\varepsilon_{ij}, \qquad (h=1, 2, \ldots, n^2-r),$$

provided these n^2 differential operators are independent, that is, provided the n^2 $\mathscr{A}$'s are linearly independent linear homogeneous functions of the ε's. In which case

$$\left| \begin{matrix} a_{i11}, a_{i12}, \ldots, a_{i1n}, a_{i21}, \ldots, a_{in1}, a_{in2}, \ldots, a_{inn} \\ i=1, 2, \ldots, n^2 \end{matrix} \right| \neq 0.$$

I shall employ the following notation. The constituent of any matrix B in the *p-th* row and *q-th* column will be denoted by $(B)_{pq}$ and the determinant of B by $|B|$. Thus, if T denote the above matrix, $|T| \neq 0$ and $(T)_{i,(\mu-1)n+\nu} = a_{i\mu\nu}$, $(i=1, 2, \ldots, n^2;\ \mu, \nu=1, 2, \ldots, n)$. We shall now have, in Cayley's matrical notation,

$$(\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n^2}) = T(\varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{1n}, \varepsilon_{21}, \ldots, \varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn}). \quad (1)$$

For $1 \leqq i \leqq n^2$ the equations of the infinitesimal transformation of G corresponding to $\mathscr{A}_i$, namely,

$$x'_\mu = x_\mu + \delta t\mathscr{A}_i x_\mu, \qquad (\mu=1, 2, \ldots, n),$$

may be written $(x'_1, x'_2, \ldots, x'_n) = (1+\delta t A_i)(x_1, x_2, \ldots, x_n)$ where A_i is the matrix corresponding to $\mathscr{A}_i$ and

$$(A_i)_{\mu\nu} = a_{i\mu\nu}, \qquad (\mu, \nu=1, 2, \ldots, n). \quad (2)$$

For our present purpose we may substitute the matrices $A_1, A_2, \ldots, A_{n^2}$ for the respective differential operators $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_{n^2}$, regarding the former as the symbols of the infinitesimal transformations of G. Similarly, if e_{ij}, $(i, j=1, 2, \ldots, n)$, denote the matrix whose constituents are all zero except that in the *i-th* row and *j-th* column which is equal to 1, we may substitute

e_{ij}, $(i, j=1, 2, \ldots, n)$, for ε_{ij}. The general infinitesimal transformation of G_r is then represented by

$$A'=\sum_{h=1}^{r}\alpha_h\mathcal{A}_h=\sum_{h=1}^{r}\sum_{i,j=1}^{n}\alpha_h a_{hij}\varepsilon_{ij}, \text{ or by the matrix } A=\sum_{h=1}^{r}\alpha_h A_h,$$

where $(A')_{\mu\nu}=\sum_{i=1}^{r}\alpha_i a_{i\mu\nu}$, $(\mu, \nu=1, 2, \ldots, n)$. The general infinitesimal transformation of the general linear homogeneous group is represented by

$$A=\sum_{h=1}^{n^2}\alpha_h\mathcal{A}_h=\sum_{h=1}^{n^2}\sum_{i,j=1}^{n}\alpha_h a_{hij}\varepsilon_{ij}, \text{ or by the matrix } A=\sum_{h=1}^{n^2}\alpha_h A_h,$$

where $(A)_{\mu\nu}=a_{\mu\nu}=\sum_{i=1}^{n^2}\alpha_i a_{i\mu\nu}$, $(\mu, \nu=1, 2, \ldots, n)$.

We shall have then

$$(A_i, A_j)=\sum_{k=1}^{r}c_{ijk}A_k, \qquad (i, j=1, 2, \ldots, r), \tag{3}$$

$$(A_i, A_j)=\sum_{k=1}^{n^2}c_{ijk}A_k, \qquad (i, j=1, 2, \ldots, n^2),$$

where c_{ijk}, $(i, j, k=1, 2, \ldots, n^2)$, are the structural constants of G, and c_{ijk}, $(i, j, k=1, 2, \ldots, r)$, are the structural constants of G_r. Since there is no linear relation with constant coefficients between $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$, the corresponding matrices $A_1, A_2, \ldots, A_r$ are linearly independent; similarly, since there is no linear relation with constant coefficients between $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n^2}$, then $A_1, A_2, \ldots, A_{n^2}$ are linearly independent.

Let Γ denote the adjoint of the general linear homogeneous group G, and Γ_r the adjoint of G_r. The symbols of the infinitesimal transformations of the adjoint are differential operators which may be replaced for our present purpose by the matrices representing the square arrays of their coefficients. Thus, if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n^2}$ are chosen as the symbols of the infinitesimal transformations of G, the corresponding symbols of the infinitesimal transformations of the adjoint are $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{n^2}$, where $\mathcal{E}_i=\sum_{j,k=1}^{n^2}c_{ijk}\alpha_j\dfrac{\partial}{\partial\alpha_k}$, $(i=1, 2, \ldots, n^2)$. The equations, $\alpha'_\mu=\alpha_\mu+\delta t\mathcal{E}_i\alpha_\mu=\alpha_\mu+\delta t\sum_{j=1}^{n^2}c_{ij\mu}\alpha_j$, $(\mu=1, 2, \ldots, n^2)$, defining the infinitesimal transformation of Γ may be written in matrical form as follows: $(\alpha'_1, \alpha'_2, \ldots, \alpha'_{n^2})=(1+\delta tE_i)(\alpha_1, \alpha_2; \ldots, \alpha_{n^2})$, where E_i is the matrix defined by the equations

$$(E_i)_{\nu\mu}=c_{i\mu\nu}, \qquad (i, \mu, \nu=1, 2, \ldots, n^2); \tag{4}$$

and we may substitute for the symbols $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{n^2}$, the respectively corresponding matrices $E_1, E_2, \ldots, E_{n^2}$. A similar remark applies to the

group Γ_r, whose symbols of infinitesimal transformation are $\mathcal{E}'_1, \mathcal{E}'_2, \ldots, \mathcal{E}'_r$, where $\mathcal{E}'_i = \sum_{j,k=1}^{r} c_{ijk}\alpha_j \frac{\partial}{\partial \alpha_k}$, $(i=1, 2, \ldots, r)$, and these may be replaced by the respectively corresponding matrices $E'_1, E'_2, \ldots, E'_r$, where

$$(E'_i)_{\nu\mu} = c_{i\mu\nu}, \qquad (i, \mu, \nu = 1, 2, \ldots, r). \tag{5}$$

We have

$$(E_i, E_j) = \sum_{k=1}^{n^2} c_{ijk} E_k, \qquad (i, j = 1, 2, \ldots, n^2),$$

$$(E'_i, E'_j) = \sum_{k=1}^{r} c_{ijk} E'_k, \qquad (i, j = 1, 2, \ldots, r).$$

When $A_1, A_2, \ldots, A_{n^2}$ are chosen as the symbols of the infinitesimal transformations of G, the determination of the structural constants c_{ijk}, and hence of the matrices $E_1, E_2, \ldots, E_{n^2}$, may be accomplished as follows. The matrices $A_1, A_2, \ldots, A_{n^2}$, constitute a hypercomplex number system. Let $A_i A_j = \sum_{k=1}^{n^2} \gamma_{ijk} A_k$; by a theorem of Lie's (*Transformationsgruppen*, Vol. III, § 140)

$$c_{ijk} = \gamma_{ijk} - \gamma_{jik}. \tag{6}$$

To find γ_{ijk} we note that $e_{\mu\nu}$, $(\mu, \nu = 1, 2, \ldots, n)$, constitute a number system equivalent to $A_1, A_2, \ldots, A_{n^2}$, with a multiplication table determined by the equations $e_{\mu\nu} e_{\nu\sigma} = e_{\mu\sigma}$ and $e_{\mu\nu} e_{\lambda\sigma} = 0$, $(\mu, \nu, \lambda, \sigma = 1, 2, \ldots, n;\ \lambda \neq \nu)$. We may put $e_{\mu\nu} e_{\lambda\sigma} = \sum_{\rho,\tau=1}^{n} g_{\mu,\nu,\lambda,\sigma,\rho,\tau}\, e_{\rho\tau}$, provided $g_{\mu,\nu,\lambda,\sigma,\rho,\tau} = 0$ for all values of the subscripts except that

$$g_{\mu,\nu,\nu,\sigma,\mu,\sigma} = 1, \qquad (\mu, \nu, \sigma = 1, 2, \ldots, n).$$

From equation (1), or rather the equation

$$(A_1, A_2, \ldots, A_{n^2}) = T(e_{11}, e_{12}, \ldots, e_{n1}, e_{n2}, \ldots, e_{nn}),$$

we derive

$$(e_{11}, e_{12}, \ldots, e_{n1}, e_{n2}, \ldots, e_{nn}) = T^{-1}(A_1, A_2, \ldots, A_{n^2}). \tag{7}$$

Then $(T)_{i,\,(\mu-1)n+\nu} = a_{i\mu\nu}$,

$$(T^{-1})_{(\mu-1)n+\nu,\,i} = (-1)^{(\mu-1)n+\nu+i} \frac{1}{|T|} \frac{\partial}{\partial a_{i\mu\nu}} |T|,$$

$$(i = 1, 2, \ldots, n^2;\ \mu, \nu = 1, 2, \ldots, n);$$

and equations (1) and (7) are respectively equivalent to the two systems of equations

$$A_i = \sum_{\mu,\nu=1}^{n} a_{i\mu\nu} e_{\mu\nu} = \sum_{\mu,\nu=1}^{n} (T)_{i,\,(\mu-1)n+\nu} e_{\mu\nu}, \qquad (i = 1, 2, \ldots, n^2),$$

$$e_{\mu\nu} = \sum_{i=1}^{n^2} (T^{-1})_{(\mu-1)n+\nu,\,i} A_i, \qquad (\mu, \nu = 1, 2, \ldots, n).$$

$$\text{Therefore, } A_iA_j=\sum_{\mu,\nu=1}^{n}(T)_{i,\,(\mu-1)n+\nu}e_{\mu\nu}\sum_{\lambda,\sigma=1}^{n}(T)_{j,\,(\lambda-1)n+\sigma}e_{\lambda\sigma}$$
$$=\sum_{\mu,\nu,\lambda,\sigma=1}^{n}(T)_{i,\,(\mu-1)n+\nu}(T)_{j,\,(\lambda-1)n+\sigma}e_{\mu\nu}e_{\lambda\sigma}$$
$$=\sum_{\mu,\nu,\lambda,\sigma,\rho,\tau=1}^{n}(T)_{i,\,(\mu-1)n+\nu}(T)_{j,\,(\lambda-1)n+\sigma}g_{\mu,\nu,\lambda,\sigma,\rho,\tau}e_{\rho,\tau}$$
$$=\sum_{\mu,\nu,\lambda,\sigma,\rho,\tau=1}^{n}(T)_{i,\,(\mu-1)n+\nu}(T)_{j,\,(\lambda-1)n+\sigma}g_{\mu,\nu,\lambda,\sigma,\rho,\tau}\sum_{k=1}^{n^2}(T^{-1})_{(\rho-1)n+\tau,k}A_k,$$
$$(i,\ j=1,\ 2,\ \ldots,\ n^2).$$

If the matrix M_i, $(1\leqq i\leqq n^2)$ of order n^2 be defined by the equations

$$(M_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}=\sum_{\mu,\nu=1}^{n}(T)_{i,\,(\mu-1)n+\nu}g_{\mu,\nu,\lambda,\sigma,\rho,\tau},\quad(\lambda,\sigma,\rho,\tau=1,2,\ldots,n),\quad(8)$$

we may write

$$A_iA_j=\sum_{k=1}^{n^2}\sum_{\rho,\tau,\lambda,\sigma=1}^{n}(T)_{j,\,(\lambda-1)n+\sigma}(M_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}(T^{-1})_{(\rho-1)n+\tau,\,k}A_k$$
$$=\sum_{k=1}^{n^2}\sum_{\lambda,\sigma,\rho,\tau=1}^{n}(\breve{T}^{-1})_{k,\,(\rho-1)n+\tau}(M_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}(\breve{T})_{(\lambda-1)n+\sigma,\,j}A_k$$
$$=\sum_{k=1}^{n^2}(\breve{T}^{-1}M_i\breve{T})_{kj}A_k,\qquad(i,\ j=1,\ 2,\ \ldots,\ n^2),\quad(9)$$

where $\breve{T}$ and $\breve{T}^{-1}$ denote respectively the transverse or conjugate of T and T^{-1}.* Therefore, since the A's are linearly independent,

$$\gamma_{ijk}=(\breve{T}^{-1}N_i\breve{T})_{kj},\qquad(i,\ j,\ k=1,\ 2,\ \ldots,\ n^2).\quad(10)$$

Interchanging A_i and A_j in (9) we obtain

$$A_jA_i=\sum_{k=1}^{n^2}\sum_{\mu,\nu,\lambda,\sigma,\rho,\tau=1}^{n}(T)_{j,\,(\lambda-1)n+\sigma}(T)_{i,\,(\mu-1)n+\nu}g_{\lambda,\sigma,\mu,\nu,\rho,\tau}(T^{-1})_{(\rho-1)n+\tau,\,j}A_k,$$
$$(i,\ j=1,\ 2,\ \ldots,\ n^2).$$

and defining the matrix N_i by the equations

$$(N_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}=\sum_{\mu,\nu=1}^{n}(T)_{i,\,(\mu-1)n+\nu}g_{\lambda,\sigma,\mu,\nu,\rho,\tau},$$
$$(\lambda,\ \sigma,\ \rho,\ \tau=1,\ 2,\ \ldots,\ n;\ i=1,\ 2,\ \ldots,\ n^2),\quad(11)$$

$$A_jA_i=\sum_{k=1}^{n^2}\sum_{\lambda,\sigma,\rho,\tau=1}^{n}(T)_{j,\,(\lambda-1)n+\sigma}(N_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}(T^{-1})_{(\rho-1)n+\tau,\,k}A_k$$
$$=\sum_{k=1}^{n^2}(\breve{T}^{-1}N_i\breve{T})_{kj}A_k,\qquad(i,\ j=1,\ 2,\ \ldots,\ n^2),$$

and therefore,

$$\gamma_{jik}=(\breve{T}^{-1}N_i\breve{T})_{kj},\qquad(i,\ j,\ k=1,\ 2,\ \ldots,\ n^2).\quad(12)$$

* The transverse of any matrix B is obtained by interchanging rows and columns so that $(B)_{ij}=b_{ij}=(\breve{B})_{ji}$. We have $(\breve{B})^{-1}=\breve{(B^{-1})}$; and if C is any second matrix of order n, $\breve{(B\pm C)}=\breve{B}\pm\breve{C}$ and $\breve{(BC)}=(\breve{C}\breve{B})$.

By virtue of the equations (6), (11), (12), we now have

$$c_{ijk}=(\breve{T}^{-1}M_i\breve{T})_{kj}-(\breve{T}^{-1}N_i\breve{T})_{kj}=(\breve{T}^{-1}(M_i-N_i)\breve{T})_{kj}, \qquad (i, j, k=1, 2, \ldots, n^2),$$

and therefore, by equation (4) which defines the matrices $E_1, E_2, \ldots, E_{n^2}$,

$$E_i=\breve{T}^{-1}(M_i-N_i)\breve{T}, \qquad (i=1, 2, \ldots, n^2). \tag{13}$$

The matrices M_i and N_i are respectively defined by the equations

$$(M_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}=\sum_{\mu,\,\nu=1}^{n} a_{i\mu\nu}g_{\mu,\nu,\lambda,\sigma,\rho,\tau}=\sum_{\mu,\,\nu=1}^{n}(A_i)_{\mu\nu}g_{\mu,\nu,\lambda,\sigma,\rho,\tau}, \tag{8}$$

$$(N_i)_{(\rho-1)n+\tau,\,(\lambda-1)n+\sigma}=\sum_{\mu,\,\nu=1}^{n} a_{i\mu\nu}g_{\lambda,\sigma,\mu,\nu,\rho,\tau}=\sum_{\mu,\,\nu=1}^{n}(A_i)_{\mu\nu}g_{\lambda,\sigma,\mu,\nu,\rho,\tau}, \tag{11}$$

$$(i=1, 2, \ldots, n^2),$$

where $g_{\mu,\nu,\lambda,\sigma,\rho,\tau}=0$ for all values of the subscripts except that $g_{\mu,\nu,\nu,\sigma,\mu,\sigma}=1$, $(\mu, \nu, \sigma=1, 2, \ldots, n)$. Whence it follows that

$$(M_i)_{(\mu-1)n+\sigma,\,(\nu-1)n+\sigma}=a_{i\mu\nu}=(A_i)_{\mu\nu}, \qquad (\mu, \nu, \sigma=1, 2, \ldots, n),$$
$$(N_i)_{(\lambda-1)n+\nu,\,(\lambda-1)n+\mu}=a_{i\mu\nu}=(A_i)_{\mu\nu}, \qquad (\lambda, \mu, \nu=1, 2, \ldots, n),$$

and that all other constituents of M_i and N_i $(1\leqq i\leqq n^2)$ are zero. Hence we find that the matrix M_i which is of order n^2, may be represented in the following way: $(M_i)_{\mu\nu}=M_{i\mu\nu}$, $(\mu, \nu=1, 2, \ldots, n)$, where, for $1\leqq i\leqq n^2$, the constituent $M_{i\mu\nu}$ is itself a matrix of order n defined as follows: $(M_{i\mu\nu})_{pp}=a_{i\mu\nu}$, $(M_{i\mu\nu})_{pq}=0$, $(p, q=1, 2, \ldots, n;\ p\neq q)$. If I denotes the identical substitution in n variables we may write

$$(M_i)_{\mu\nu}=(A_i)_{\mu\nu}I, \qquad (\mu, \nu=1, 2, \ldots, n;\ i=1, 2, \ldots, n^2). \tag{14}$$

Similarly, we find that N_i can be expressed in terms of constituent matrices as follows:

$$(N_i)_{pp}=\breve{A}_i,\ (N_i)_{pq}=0, \qquad (p, q=1, 2, \ldots, n;\ p\neq q;\ i=1, 2, \ldots, n^2). \tag{15}$$

If B, C, Q and R are any four matrices of order n, for two matrices W_b and W_c such that $(W_b)_{\mu\nu}=(B)_{\mu\nu}Q$ and $(W_c)_{\mu\nu}=(C)_{\mu\nu}R$, $(\mu, \nu=1, 2, \ldots, n)$, we have

$$(W_bW_c)_{\mu\nu}=\sum_{h=1}^{n}(B)_{\mu h}(C)_{h\nu}QR^{*}=(BC)_{\mu\nu}QR=(W_{bc})_{\mu\nu}, \quad (\mu, \nu=1, 2, \ldots, n). \tag{16}$$

By this formula we obtain

$$(M_iM_j)_{\mu\nu}=(A_iA_j)_{\mu\nu}I, \tag{17}$$

$$(M_i^2)_{\mu\nu}=(A_i^2)_{\mu\nu}I, \quad (M_i^3)_{\mu\nu}=(M_iM_i^2)_{\mu\nu}=(A_i^3)_{\mu\nu}I, \tag{18}$$

and for any positive integer p

$$(M_i^p)_{\mu\nu}=(A_i^p)_{\mu\nu}I, \qquad (\mu, \nu=1, 2, \ldots, n;\ i, j=1, 2, \ldots, n^2); \tag{19}$$

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* The law of multiplication for matrices whose constituents are themselves matrices is the same as for matrices whose constituents are scalars.

again from (15) we have,

$$\left.\begin{aligned}(N_iN_j)_{\mu\nu}&=\sum_{h=1}^{n}(N_i)_{\mu h}(N_j)_{h\nu}=0,\\ (N_iN_j)_{\mu\mu}&=\sum_{h=1}^{n}(N_i)_{\mu h}(N_j)_{h\mu}=(N_i)_{\mu\mu}(N_j)_{\mu\mu}=\breve{A}_i\breve{A}_j,\end{aligned}\right\}\quad(20)$$

and for any positive integer p

$$(N_i^p)_{\mu\mu}=\breve{A}_i^p,\quad (N_i^p)_{\mu\nu}=0,\qquad (\mu,\nu=1,2,\ldots,n;\ \mu\neq\nu;\ i,j=1,2,\ldots,n^2).\quad(21)$$

Further, by equations (16), (19) and (21) for positive integers p, q, r, s, t, v we obtain

$$\begin{aligned}(M_i^sN_j^p)_{\mu\nu}&=\sum_{h=1}^{n}(M_i^s)_{\mu h}(N_j^p)_{h\nu}=(M_i^s)_{\mu\nu}(N_j^p)_{\nu\nu}=(A_i^s)_{\mu\nu}I\breve{A}_j^p=(N_j^p)_{\mu\mu}(M_i^s)_{\mu\nu}\\ &=\sum_{h=1}^{n}(N_j^p)_{\mu h}(M_i^s)_{h\nu}=(N_j^pM_i^s)_{\mu\nu},\end{aligned}\quad(22)$$

$$(M_i^pN_i^qM_j^rN_j^s)_{\mu\nu}=((M_i^pN_i^q)(M_j^rN_j^s))_{\mu\nu}=(A_i^pA_j^r)_{\mu\nu}\breve{A}_i^q\breve{A}_j^s,\quad(23)$$

$$(M_i^pN_i^qM_j^rN_j^sM_k^tN_k^v)_{\mu\nu}=((M_i^pN_i^qM_j^rN_j^s)(M_k^tN_k^v))_{\mu\nu}=(A_i^pA_j^rA_k^t)_{\mu\nu}\breve{A}_i^q\breve{A}_j^s\breve{A}_k^v,$$
$$(\mu,\nu=1,2,\ldots,n;\ i,j,k=1,2,\ldots,n^2).\quad(24)$$

I shall employ the symbol S prefixed to a matrix M whose constituents are scalars, designating SM as the *scalar* of M, to denote the sum of the constituents in the principal diagonal of M. Then, if A and B are any two matrices,

$$S(A\pm B)=SA\pm SB,\quad SAB=SBA,\quad S\breve{A}=SA.\quad(25)$$

Further, if ρ is any scalar

$$S\rho A=\rho SA;\quad(26)$$

and if I is the identical substitution in n variables,

$$SI=n.\quad(27)$$

Moreover, SM is the sum of the roots of the characteristic equation of M, $|\omega-M|=0$, and SM^p is the sum of the *p-th* powers of the roots. In the case of a matrix $(M)_{\mu\nu}=M_{\mu\nu}$, $(\mu,\nu=1,2,\ldots)$, whose constituents, the M's with double subscripts, are themselves matrices, those in the principal diagonal being square matrices, I shall define SM as follows:

$$SM=SM_{11}+SM_{22}+\ldots.$$

It may be readily seen that the properties of the scalar function given in equations (25), (26) and (27) hold in this case also.

We then have from equation (13), $SE_i = S\breve{T}^{-1}(M_i - N_i)\breve{T}$, and since the scalar product of two or more matrices is unchanged by an interchange of the factors leaving the cyclic order unaltered,

$$SE_i = S(M_i - N_i) = SM_i - SN_i, \qquad (i = 1, 2, \ldots, n^2).$$

By equation (14), we have $SM_i = \sum_{\mu=1}^{n} (A_i)_{\mu\mu} SI = SA_i \cdot SI = nSA_i$, and by equation (15), $SN_i = nS\breve{A}_i = nSA_i$. Substituting these results in the above we obtain

$$SE_i = 0, \qquad (i = 1, 2, \ldots, n^2). \tag{28}$$

Also

$$\begin{aligned} SE_iE_j &= S\breve{T}^{-1}(M_i - N_i)\breve{T} \cdot \breve{T}^{-1}(M_j - N_j)\breve{T} = S(M_i - N_i)(M_j - N_j) \\ &= SM_iM_j - SM_iN_j - SN_iM_j + SN_iN_j, \qquad (i, j = 1, 2, \ldots, n^2). \end{aligned} \tag{29}$$

From equation (17) we obtain $SM_iM_j = \sum_{\mu=1}^{n} (A_iA_j)_{\mu\mu} SI = SA_iA_jSI = nSA_iA_j$; from equation (20), $SN_iN_j = nS\breve{A}_i\breve{A}_j = nSA_iA_j$; from equation (22), $SM_iN_j = SN_jM_i = \sum_{\mu=1}^{n} (A_i)_{\mu\mu} S\breve{A}_j = SA_iS\breve{A}_j = SA_iSA_j$, and substituting these values in equation (29) we find

$$SE_iE_j = 2(nSA_iA_j - SA_iSA_j), \qquad (i, j = 1, 2, \ldots, n^2). \tag{30}$$

For any positive integer h we have

$$SE_i^h = (S\breve{T}^{-1}(M_i - N_i)\breve{T})^h = S(M_i - N_i)^h = \sum_{\mu=0}^{h} (-1)^\mu \binom{h}{\mu} SM_i^{h-\mu}N_i^\mu,$$
$$(i = 1, 2, \ldots, n^2).$$

Since for any odd integer, $h = 2p+1$, the number of terms in the expansion is even and the even terms have a negative sign, and the binomial coefficients $\binom{h}{j}$ and $\binom{h}{h-j}$ are equal, we may write

$$\begin{aligned} SE_i^{2p+1} &= \sum_{\mu=0}^{2p+1} (-1)^\mu \binom{2p+1}{\mu} SM_i^{2p-\mu+1}N_i^\mu \\ &= \sum_{\mu=0}^{p} (-1)^\mu \binom{2p+1}{\mu} (SM_i^{2p-\mu+1}N_i^\mu - SM_i^\mu N_i^{2p-\mu+1}), \\ &\qquad (i = 1, 2, \ldots, n^2). \end{aligned} \tag{31}$$

From equations (19), (21) and (22), respectively, we obtain

$$SM_i^q = \sum_{\mu=1}^{n} (A_i^q)_{\mu\mu} SI = SA_i^q SI = nSA_i^q,$$

$$SN_i^q = nS\breve{A}_i^q = nSA_i^q, \quad SM_i^q N_i^r = \sum_{\mu=1}^{n} (A_i^q)_{\mu\mu} S\breve{A}_i^r = SA_i^q SA_i^r = SM_i^r N_i^q,$$

and substituting in equation (31)

$$SE_i^{2p+1}=0, \qquad (i=1, 2, \ldots, n^2). \tag{32}$$

For any even integer, $h=2p$, we have

$$SE_i^{2p}=\sum_{\mu=0}^{2p}(-1)^\mu\binom{2p}{\mu}SA_i^{2p-\mu}SA_i^\mu, \qquad (i=1, 2, \ldots, n^2), \tag{33}$$

where SA_i^0 is assigned the value n. For any positive integers p and q

$$SE_i^pE_j^q=S(\breve{T}^{-1}(M_i-N_i)\breve{T})^p(\breve{T}^{-1}(M_j-N_j)\breve{T})^q=S(M_i-N_i)^p(M_j-N_j)^q$$
$$=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}(-1)^{\mu+\nu}\binom{p}{\mu}\binom{q}{\nu}SM_i^{p-\mu}N_i^\mu M_j^{q-\nu}N_j^\nu, \ (i, j=1, 2, \ldots, n^2). \tag{34}$$

From (23) we obtain $SM_i^{p-\mu}N_i^\mu M_j^{q-\nu}N_j^\nu=SA_i^{p-\mu}A_j^{q-\nu}S\breve{A}_i^\mu\breve{A}_j^\nu=SA_i^{p-\mu}A_j^{q-\nu}SA_i^\mu A_j^\nu$, and substituting in (34),

$$SE_i^pE_j^q=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}(-1)^{\mu+\nu}\binom{p}{\mu}\binom{q}{\nu}SA_i^{p-\mu}A_j^{q-\nu}SA_i^\mu A_j^\nu, \quad (i, j=1, 2, \ldots, n^2), \tag{35}$$

where $A_i^0=A_j^0=I$. Again, for positive integers p, q and r

$$SE_i^pE_j^qE_k^r=S(M_i-N_i)^p(M_j-N_j)^q(M_k-N_k)^r$$
$$=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}\sum_{\sigma=0}^{r}(-1)^{\mu+\nu+\sigma}\binom{p}{\mu}\binom{q}{\nu}\binom{r}{\sigma}SM_i^{p-\mu}N_i^\mu M_j^{q-\nu}N_j^\nu M_k^{r-\sigma}N_k^\sigma,$$
$$(i, j, k=1, 2, \ldots, n^2). \tag{36}$$

From equation (24) $SM_i^{p-\mu}N_i^\mu M_j^{q-\nu}N_j^\nu M_k^{r-\sigma}N_k^\sigma=SA_i^{p-\mu}A_j^{q-\nu}A_k^{r-\sigma}S\breve{A}_i^\mu\breve{A}_j^\nu\breve{A}_k^\sigma$, and substituting in (36) we have

$$SE_i^pE_j^qE_k^r=\sum_{\mu=0}^{p}\sum_{\nu=0}^{q}\sum_{\sigma=0}^{r}(-1)^{\mu+\nu+\sigma}\binom{p}{\mu}\binom{q}{\nu}\binom{r}{\sigma}SA_i^{p-\mu}A_j^{q-\nu}A_k^{r-\sigma}S\breve{A}_i^\mu\breve{A}_j^\nu\breve{A}_k^\sigma,$$
$$(i, j, k=1, 2, \ldots, n^2), \tag{37}$$

where $A_i^0=A_j^0=A_k^0=I$. In particular

$$SE_iE_jE_k=n(SA_iA_jA_k-S\breve{A}_i\breve{A}_j\breve{A}_k)=nS(A_i, A_j)A_k,$$
$$(i, j, k=1, 2, \ldots, n^2). \tag{38}$$

The equations (32) and (33) enable us to establish certain relations between the roots of the characteristic equation,

$$|\omega-E_i|=\omega^{n^2}-\Psi_1\omega^{n^2-1}+\Psi_2\omega^{n^2-2}-\ldots\pm\Psi_{n^2-1}\omega\mp\Psi_{n^2}=0, \quad (\Psi_{n^2}=0),*$$

of E_i and the roots of the characteristic equation,

$$|\rho-A_i|=\rho^n-P_1\rho^{n-1}+P_2\rho^{n-2}-\ldots\mp P_{n-1}\rho\pm P_n=0,$$

of A_i. Let $\omega_{11}, \omega_{12}, \ldots, \omega_{nn}$ denote the roots of the equation $|\omega-E_i|=0$, and $\mathcal{S}_q$ the sum of the q-th powers of the roots; let $\rho_1, \rho_2, \ldots, \rho_n$ denote the roots

* Killing has shown that the characteristic equation of the general infinitesimal transformation of the adjoint has at least one zero root (*Math. Ann.*, Vol. XXXI, p. 260).

of $|\rho-A_i|=0$, and s_q the sum of their *q-th* powers. By equation (32) $\mathscr{S}_{2p+1} = SE_i^{2p+1}=0$, $(p=0, 1, 2, \ldots.)$, wherefore $\Psi_{2p+1}=0$, $(p=0, 1, 2, \ldots.)$, and the non-zero roots of $|\omega-E_i|=0$ occur in pairs of equal value but opposite sign;* from equation (33)

$$\mathscr{S}_{2p}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}s_{2p-\mu}s_{\mu}, \qquad (p=0, 1, 2, \ldots.),$$

from which relation it then follows that the non-zero roots of $|\omega-E_i|=0$ not exceeding n^2-n in number, shall be related to the roots of $|\rho-A_i|=0$ by the equation $\omega_{ij}=\rho_i-\rho_j$, $(i \neq j;\ i, j=1, 2, \ldots., n)$.†

For any linear function of the E's with constant coefficients, $\sum_{i=1}^{n^2}\alpha_i E_i$, we have

$$E=\sum_{i=1}^{n^2}\alpha_i E_i=\breve{T}^{-1}(M-N)\breve{T}, \tag{39}$$

where $M=\sum_{i=1}^{n^2}\alpha_i M_i$ and $N=\sum_{i=1}^{n^2}\alpha_i N_i$. If $A=\sum_{i=1}^{n^2}\alpha_i A_i$ we see from equation (14) that $(M)_{\mu\nu}=(A)_{\mu\nu}I$, $(\mu, \nu=1, 2, \ldots., n)$, and by the aid of equation (16) we can show that for any positive integer p, $(M^p)_{\mu\nu}=(A^p)_{\mu\nu}I$, $(\mu, \nu=1, 2, \ldots., n)$. It is evident from equation (15) that $(N)_{\mu\mu}=\breve{A}$ and $(N^p)_{\mu\mu}=\breve{A}^p$ for any positive integer p $(\mu=1, 2, \ldots., n)$ and that all other constituent matrices of these matrices are zero. Also corresponding to equation (22) we have $(M^pN^q)_{\mu\nu}=(A^p)_{\mu\nu}\breve{A}^q=(N^qM^p)_{\mu\nu}$, $(\mu, \nu=1, 2, \ldots., n)$ for any positive integers p and q. Hence

$$\left.\begin{aligned} SM^p&=\sum_{\mu=1}^{n}(A^p)_{\mu\mu}SI=nSA^p, \quad SN^p=\sum_{\mu=1}^{n}(N^p)_{\mu\mu}=nSA^p,\\ SM^pN^q&=\sum_{\mu=1}^{n}(A^p)_{\mu\mu}S\breve{A}^q=SA^pSA^q=SM^qN^p. \end{aligned}\right\} \tag{40}$$

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* Substituting these values of $\mathscr{S}_{2p+1}$ in Newton's formula,

$$\mathscr{S}_k-\Psi_1\mathscr{S}_{k-1}+\Psi_2\mathscr{S}_{k-2}-\ldots.+(-1)^{k-1}\Psi_{k-1}\mathscr{S}_1+(-1)^k k\Psi_k=0,$$

and considering only $k=1, 3, 5, \ldots.$, we obtain $\Psi_{2p+1}=0$ for $p=0, 1, 2, \ldots.$.

† Let $\mathscr{S}_h$ be the sum of the *h-th* powers of the $n(n-1)$ non-zero roots $\rho_i-\rho_j$, $(i, j=1, 2, \ldots., n;\ i \neq j)$. Each root is paired with its negative and hence $\mathscr{S}_{2p+1}=0$, $(p=1, 2, \ldots.)$. Let

$$\phi(\rho)=\sum_{i=1}^{n}(\rho-\rho_i)^{2p}=\sum_{i=1}^{n}\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}\rho^{2p-\mu}\rho_i^{\mu}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}\rho^{2p-\mu}s_{\mu}, \quad (p=1, 2, \ldots.).$$

Then $\mathscr{S}_{2p}=\sum_{j=1}^{n}\phi(\rho_j)=\sum_{j=1}^{n}\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}\rho_j^{2p-\mu}s_{\mu}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}s_{2p-\mu}s_{\mu}$, $(p=1, 2, \ldots.)$.

From equation (39) we obtain

$$SE^{2p+1}=S(\check{T}^{-1}(M-N)\check{T})^{2p+1}=S(M-N)^{2p+1}$$
$$=\sum_{\mu=0}^{2p+1}(-1)^{\mu}\binom{2p+1}{\mu}SM^{2p-\mu+1}N^{\mu}$$
$$=\sum_{\mu=0}^{p}(-1)^{\mu}\binom{2p+1}{\mu}(SM^{2p-\mu+1}N^{\mu}-SM^{\mu}N^{2p-\mu+1}),\quad (p=0,1,2,\ldots.),$$

and $SE^{2p}=S(M-N)^{2p}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}SM^{2p-\mu}N^{\mu}$, $(p=1,2,\ldots.)$, and substituting from equations (40) we have

$$SE^{2p+1}=0,\quad SE^{2p}=\sum_{\mu=0}^{2p}(-1)^{\mu}\binom{2p}{\mu}SA^{2p-\mu}SA^{\mu},\qquad (p=1,2,\ldots.).\quad (41)$$

The characteristic equation of E, where $E=\sum_{i=1}^{n^2}\alpha_i E_i$ for an arbitrary choice of the α's, is termed by Killing the "characteristic equation" of the group G. The characteristic equation of $\sum_{i=1}^{r}\alpha_i E_i$, for an arbitrary choice of the α's, is termed by Cartan the "characteristic equation" of G relative to the subgroup G_r. It follows from equations (41) that, for $1\overline{\overline{<}}r\overline{\overline{<}}n^2$, the roots of $|\omega-\sum_{i=1}^{r}\alpha_i E_i|=0$ are the n^2 differences of the roots of the equation $|\rho-\sum_{i=1}^{r}\alpha_i A_i|=0$, and hence we have the following theorem:

The roots of the characteristic equation of the general linear homogeneous group relative to any given subgroup can be obtained by taking all possible differences of the roots of the characteristic equation of the general infinitesimal transformation of this subgroup of the general linear homogeneous group. The non-zero roots do not exceed n^2-n in number and occur in pairs of equal absolute value but opposite sign.

Section 2.

The special linear homogeneous group in n variables, denoted in what follows by G', is constituted by the aggregate of transformations with determinant $+1$ of the general linear homogeneous group, G, in n variables. I shall now assume that G_r is also a subgroup of the special linear homogeneous group, G'. For $1\leqq i\leqq r$ let $\rho_1,\rho_2,\ldots.,\rho_n$ be the roots of the characteristic equation, $|\rho-A_i|=0$, of A_i. The finite transformations of G' are given by e^{tA_i} for all possible values of t. The roots of the characteristic equation of e^{tA_i} are $e^{t\rho_1}, e^{t\rho_2},\ldots., e^{t\rho_n}$; and since $e^{t(\rho_1+\rho_2+\ldots.+\rho_n)}=|e^{tA_i}|=1$ for all values of t, $\rho_1+\rho_2+\ldots.+\rho_n=0$. Therefore, $SA_i=0$, $(i=1,2,\ldots.,r)$. Therefore, $S\sum_{i=1}^{r}a_iA_i=\sum_{i=1}^{r}a_iSA_i=0$.

We may take as the symbols of the infinitesimal transformations of G', any n^2-1 linearly independent linear homogeneous functions of the differential operators ε_{ij}, $(i, j=1, 2, \ldots, n;\ i \neq j)$, and $\varepsilon_{ii}-\varepsilon_{nn}$, $(i=1, 2, \ldots, n-1)$, or of the aggregate of respectively corresponding matrices. Thus we may take as the symbols of the infinitesimal transformations of G', $A_1, A_2, \ldots, A_r$ together with any n^2-r-1 other matrices linear in those above mentioned, the n^2-1 A's defined by equation (1), being linearly independent and subject to the condition $\sum_{\mu=1}^{n} a_{i\mu\mu}=SA_i=0$, $(i=1, 2, \ldots, n^2-1)$. For the symbols of the infinitesimal transformations of G we may take the n^2-1 linearly independent linear homogeneous functions of ε_{ij}, $(i, j=1, 2, \ldots, n;\ i\neq j)$, and $\varepsilon_{ii}-\varepsilon_{nn}$, $(i=1, 2, \ldots, n-1)$, which constitute the symbols of the infinitesimal transformations of G' together with the operator $\sum_{\mu=1}^{n} \varepsilon_{\mu\mu}$, or the aggregate of matrices corresponding to these; and therefore, we may choose as the aggregate of matrices representing the infinitesimal transformations of G the matrices $A_1, A_2, \ldots, A_{n^2-1}$ and $A_{n^2}=\sum_{\mu=1}^{n} e_{\mu\mu}=I$.

Since G' is an invariant subgroup of G, we have

$$(A_i, A_j)=\sum_{k=1}^{n^2-1} c_{ijk}A_k, \qquad (i, j=1, 2, \ldots, n^2-1);$$

moreover, $(A_i, A_{n^2})=0$, $(i=1, 2, \ldots, n^2)$, and therefore,

$$c_{ijk}=0, \qquad (i, j \text{ or } k=n^2). \tag{42}$$

Let Γ' denote the adjoint group of the special linear homogeneous group and let the symbols of the infinitesimal transformations be the differential operators $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{n^2-1}$ where

$$\mathcal{H}_i=\sum_{j=1}^{n^2-1} c_{ijk}\alpha_j \frac{\partial}{\partial\alpha_k}, \qquad (i=1, 2, \ldots, n^2-1),$$

or the respectively corresponding matrices $H_1, H_2, \ldots, H_{n^2-1}$ where

$$(H_i)_{\nu\mu}=c_{i\mu\nu}, \qquad (\mu, \nu, i=1, 2, \ldots, n^2-1). \tag{43}$$

With the infinitesimal transformations of G chosen as above, we see from (4), (42) and (43) that the aggregate of matrices representing the infinitesimal transformations of Γ have the following form:

$$E_i=\begin{pmatrix} H_i & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{n^2}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (i=1, 2, \ldots, n^2-1). \tag{44}$$

Then since Γ and Γ' have the same structure as G and G', respectively,

$$(H_i, H_j) = \sum_{k=1}^{n^2-1} c_{ijk} H_k, \quad (E_i, E_j) = \sum_{k=1}^{n^2-1} c_{ijk} E_k, \qquad (i, j=1, 2, \ldots, n^2-1),$$

$$(E_i, E_{n^2}) = 0, \qquad (i=1, 2, \ldots, n^2).$$

By the aid of equation (44) we obtain the relations:

$$SE_i = SH_i, \quad SE_iE_j = SH_iH_j, \qquad (i, j=1, 2, \ldots, n^2-1), \tag{45}$$

and for any natural number p

$$SE_i^p = SH_i^p, \qquad (i=1, 2, \ldots, n^2-1). \tag{46}$$

It now follows from equations (28) and (45) that

$$SH_i = 0, \qquad (i=1, 2, \ldots, n^2-1);$$

and since $SA_i = 0$, $(i=1, 2, \ldots, n^2-1)$, we obtain from equations (30) and (45)

$$SH_iH_j = 2nSA_iA_j, \qquad (i, j=1, 2, \ldots, n^2-1). \tag{47}$$

Further, we obtain for $p=0, 1, 2, \ldots$ from equations (32), (33) and (46),

$$SH_i^{2p+1} = 0, \quad SH_i^{2p} = \sum_{\mu=0}^{2p} (-1)^\mu \binom{2p}{\mu} SA_i^{2p-\mu} SA_i^\mu, \qquad (i=1, 2, \ldots, n^2-1). \tag{48}$$

We can thereby show that the non-zero roots of the characteristic equation of H_i, $|\omega - H_i| = 0$, occur in pairs of equal value but opposite sign which can be obtained by taking all possible differences of the roots of the equation $|\rho - A_i| = 0$ for $1 \leqq i \leqq n^2-1$.

From equation (44) it follows that for any linear function of the H's with constant coefficients, $H = \sum_{i=1}^{n^2-1} \alpha_i H_i$,

$$\sum_{i=1}^{n^2-1} \alpha_i E_i = \begin{pmatrix} H, & 0 \\ 0, & 0 \end{pmatrix}. \quad \text{Hence } \left(\sum_{i=1}^{n^2-1} \alpha_i E_i\right)^p = \begin{pmatrix} H^p, & 0 \\ 0, & 0 \end{pmatrix} \text{ and } S\left(\sum_{i=1}^{n^2-1} \alpha_i E_i\right)^p = SH^p$$

for any positive integer p. From this result and equations (41) we see that $SH^{2p+1} = 0$, $SH^{2p} = \sum_{\mu=0}^{2p} (-1)^\mu \binom{2p}{\mu} SA^{2p-\mu} SA^\mu$, $(p=1, 2, \ldots)$, where $A = \sum_{i=1}^{n^2-1} \alpha_i A_i$. Therefore, the roots of $|\omega - \sum_{i=1}^{n^2-1} \alpha_i H_i| = 0$ are related to the roots of $|\rho - \sum_{i=1}^{n^2-1} \alpha_i A_i| = 0$ in the way described above. For an arbitrary choice of the α's $|\omega - \sum_{i=1}^{r} \alpha_i H_i| = 0$ is the characteristic equation of the special linear homogeneous group relative to the subgroup G_r—when $r = n^2-1$, it is the characteristic equation of the special linear homogeneous group—and $\sum_{i=1}^{r} \alpha_i A_i$ is the

general infinitesimal transformation of the group. We have then the following theorem:

The non-zero roots, not exceeding $n(n-1)$ in number, of the characteristic equation of the special linear homogeneous group in n variables relative to any given subgroup occur in pairs, equal in absolute value but opposite in sign, and can be obtained by taking all possible differences of the roots of the characteristic equation of the general infinitesimal transformation of this given subgroup.

Section 3.

The above investigation of the relations between the scalar function of the products of powers of the infinitesimal transformations of the general linear homogeneous group G in n variables (or the special linear homogeneous group G' in n variables) and the adjoint of G (or G') may be applied to establish certain relations between the characteristic equation of the general infinitesimal transformation of the subgroup G_r with r parameters of G (or G') and the characteristic equation of the general infinitesimal transformation of the adjoint of G_r, termed by Lie and Killing the "characteristic equation" of this group, Killing has taken the characteristic equation of a group as the basis of his investigations upon the structure of finite continuous groups.

As in § 1, let $A_1, A_2, \ldots, A_r$ be the infinitesimal transformations of any given subgroup G_r, of the general linear homogeneous group, and $A_1, A_2, \ldots, A_r, A_{r+1}, \ldots, A_{n^2}$ the infinitesimal transformations of G; and if G_r is a subgroup of G', let $A_1, A_2, \ldots, A_r, A_{r+1}, \ldots, A_{n^2-1}$ be the infinitesimal transformations of G'. Then, see equation (3),

$$c_{ijr+h}=0, \qquad (i, j=1, 2, \ldots, r;\ h=1, 2, \ldots, n^2-r-1 \text{ or } n^2-r^*).$$

Therefore, by (4) and (5),

$$E_i = \begin{pmatrix} E'_i, & K_i \\ 0, & F_i \end{pmatrix}, \qquad (i=1, 2, \ldots, r),$$

where $(K_i)_{\nu\mu}=c_{i,\,r+\nu,\,\mu}, \quad (\mu=1, 2, \ldots, n;\ \nu=1, 2, \ldots, n^2-r-1 \text{ or } n^2-r^*),$
$(F_i)_{\nu\mu}=c_{i,\,r+\nu,\,r+\mu}, \quad (\mu, \nu=1, 2, \ldots, n^2-r-1 \text{ or } n^2-r^*),$

and 0 denotes a matrix with n^2-r-1 or n^2-r^* rows and r columns whose constituents are all zero.

The transformations $E_1, E_2, \ldots, E_r$ constitute the infinitesimal transformations of the adjoint of G' (or of G), relative to the subgroup G_r, and

* According as G_r is or is not a subgroup of G'.

$E'_1, E'_2, \ldots, E'_r$ the infinitesimal transformations of the adjoint of G_r. The general infinitesimal transformation of the adjoint of G' (or of G) relative to G_r is $\sum_{i=1}^{r} \alpha_i E_i$ and the general infinitesimal transformation of the adjoint of G_r is $\sum_{i=1}^{r} \alpha_i E'_i$. We have $\sum_{i=1}^{r} \alpha_i E_i = \begin{pmatrix} \sum_{i=1}^{r} \alpha_i E'_i, & \sum_{i=1}^{r} \alpha_i K_i \\ 0, & \sum_{i=1}^{r} \alpha_i F_i \end{pmatrix}$ and therefore,

$$|\omega - \sum_{i=1}^{r} \alpha_i E_i| = |\omega - \sum_{i=1}^{r} \alpha_i E'_i| \cdot |\omega - \sum_{i=1}^{r} \alpha_i F_i|. \tag{49}$$

The equation

$$|\omega - \sum_{i=1}^{r} \alpha_i E_i| = \omega^{\mu} - \Psi_1(\alpha)\omega^{\mu-1} + \Psi_2(\alpha)\omega^{\mu-2} - \ldots \pm \Psi_{\mu-1}(\alpha)\omega \mp \Psi_{\mu}(\alpha) = 0,$$

in which $\Psi_{\mu}(\alpha) = 0$ * is termed by Cartan the characteristic equation of G' (or of G †) relative to the subgroup G_r of G' (or of G), and the left-hand member of this equation the "characteristic determinant" of G' (or G) relative to G_r.† The equation of the general infinitesimal transformation of the adjoint of G_r,

$$|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = \omega^{r} - \Psi_1(\alpha)\omega^{r-1} + \Psi_2(\alpha)\omega^{r-2} - \ldots \mp \Psi_{r-1}(\alpha)\omega \pm \Psi_{r}(\alpha) = 0,$$

in which $\Psi_r(\alpha) = 0$,‡ is termed by Lie and Killing the "characteristic equation" of the group G_r, and the left-hand member of this equation Cartan terms the "characteristic determinant" of the group G_r. The characteristics of the equation $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ determine in part the structure of G_r.

Killing has shown that the coefficients of the characteristic equation $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ of the group are invariants of the adjoint, and has founded a classification of groups upon the properties of these coefficients. He defines the "rank" (*Rang*) of the group as the number of the coefficients of the characteristic equation of the group that are independent (*Math. Ann.*, Vol. XXXI, p. 254). As remarked by Cartan the rank of a group is the number of roots of the characteristic equation of the group that are independent (*Thèses:* "Sur la Structure des Groupes de Transformations finis et continus," p. 28). The rank is zero if and only if the coefficients of the characteristic equation are all zero identically.

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* See note p. 438.

† According as G_r is or is not a subgroup of G'.

‡ See note p. 438.

An important classification of finite continuous groups due to Lie consists in the division into integrable and non-integrable groups. An integrable group is a group some one of whose derived groups contains only the identical transformation ("Continuierliche Gruppen," p. 58). Finite continuous groups are also classified as "simple" and "non-simple" groups. A "simple" group is a group that contains no invariant subgroup. A group is "semi-simple" if no invariant subgroup is integrable (Cartan, *Thèses*, p. 51).

We have the following theorems due to Lie, Engel, Killing and Cartan:

(i). Groups of rank zero are integrable and the derived groups of an integrable group are of rank zero (Umlauf, *These*, Leipzig).

(ii). If the first derived of a group is integrable, then the group itself is integrable (Lie, "Continuierliche Gruppen," p. 547).

(iii). The necessary and sufficient condition that a group of order r be integrable is that the transformations of its first derived annul identically the coefficient $\Psi_2(\alpha)$ in its characteristic equation (Cartan, *Thèses*, p. 47).

(iv). If an infinitesimal transformation annuls $\Psi_{r-1}(\alpha)$ without annulling all the first minors of $\sum_{i=1}^{r} \alpha_i E_i$, it annuls all the invariants of the adjoint which are homogeneous and of degree non-zero (Cartan, *Thèses*, p. 31).

(v). The characteristic equation of a simple or semi-simple group with r parameters and of rank k has $r-k$ non-zero unequal roots which occur in pairs of equal value but opposite sign, and the transformations belonging to such a pair of roots are not commutable (Killing, *Math. Ann.*, Vol. XXXIV, p. 102).

Let $\rho_1, \rho_2, \ldots, \rho_n$ be the roots of the characteristic equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ of the general infinitesimal transformation of G_r. It was shown in § 1 and § 2 that the roots of the characteristic equation $|\omega - \sum_{i=1}^{r} \alpha_i E_i| = 0$ of G' (or G) relative to G_r are given by $\rho_i - \rho_j$ for $i, j = 1, 2, \ldots, n$. Whence it follows by (49) that the roots of the characteristic equation $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ of G_r are numbers contained in the aggregate of differences of the roots of the characteristic equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ of the general infinitesimal transformation of G_r.

I shall denote by l in what follows the maximum number of distinct roots of the equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ for any infinitesimal transformation of G_r.

Let $\rho_1, \rho_2, \ldots, \rho_l$ be the distinct roots of the equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ relative to the general infinitesimal transformation of G_r. The distinct non-zero roots of $|\omega - \sum_{i=1}^{r} \alpha_i E_i| = 0$ corresponding to the general infinitesimal transformation of G' (or G) relative to the subgroup G_r and, therefore, the distinct non-zero roots of $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ relative to the general infinitesimal transformation of G_r are contained in the aggregate of $l^2 - l$ functions $\rho_i - \rho_j$ of the α's for $i, j = 1, 2, \ldots, l$ and $j \neq i$. But these $l^2 - l$ functions of the α's are themselves functions of the $l-1$ differences $\rho_i - \rho_l$, $(i = 1, 2, \ldots, l-1)$. Therefore, the non-zero roots of $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ relative to the general infinitesimal transformation of G_r are functions of at most $l-1$ functions of the α's, and thus at most $l-1$ of the roots of the equation $|\omega - \sum_{i=1}^{r} \alpha_i E_i| = 0$ are independent functions of the α's. We have, therefore, the following theorems:

I. *If the maximum number of distinct roots of the characteristic equation of the general infinitesimal transformation of G_r is l, the rank of G_r can not exceed $l-1$.*

II. *If the roots of the characteristic equation of the general infinitesimal transformation of G_r are all equal the rank of G_r is zero; therefore, G_r is integrable.*

III. *If the roots of the characteristic equation of the general infinitesimal transformation of the first or any subsequent derived group of G_r are all equal, the rank of the derived group is zero; therefore, G_r is integrable.*

The roots of the characteristic equation of any infinitesimal transformation of the following group

$$A_1 = x_3 \frac{\partial}{\partial x_1}, \quad A_2 = x_2 \frac{\partial}{\partial x_1}, \quad A_3 = x_3 \frac{\partial}{\partial x_2}, \quad A_4 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$$

are all equal; this group illustrates theorems II and III, for the characteristic equation of the general infinitesimal transformation is

$$|\rho - \sum_{i=1}^{4} \alpha_i A_i| = (\rho - \alpha_4)^3 = 0.$$

The first derived of this group is $A_1 = x_3 \frac{\partial}{\partial x_1}$. The group contains two exceptional infinitesimal transformations, viz.,

$$A_1 = x_3 \frac{\partial}{\partial x_1}, \quad A_4 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}.$$

From what was stated above it is clear that if $l' \leqq l$ is the number of independent roots of the equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$, or the number of coefficients of this equation which are independent functions of the α's, the number of independent roots of the equation $|\omega - \sum_{i=1}^{r} \alpha_i E'_i| = 0$ can not exceed l'. Whence it follows that:

IV. *If l' is the number of independent coefficients of the characteristic equation*

$$|\rho - \sum_{i=1}^{r} \alpha_i A_i| = \rho^n - P_1 \rho^{n-1} + \ldots . \mp P_{n-1}\rho \pm P_n = 0,$$

of the general infinitesimal transformation G_r, the rank of G_r can not exceed l'.

The maximum number of distinct roots of the equation $|\rho - \sum_{i=1}^{r} \alpha_i A_i| = 0$ for any group G_r in n variables is n. Therefore, from I follows the theorem:

V. *The rank of no subgroup of the general linear homogeneous group in n variables can exceed $n-1$.*

By the preceding theorem the rank of the special linear homogeneous group, G', can not exceed $n-1$. Suppose the rank of G' to be $k < n-1$. G' is a simple group (*Transformationsgruppen*, Vol. I, p. 560); and hence by theorem (v) the number of non-zero roots of the characteristic equation of G' is $n^2 - k - 1$. If $k < n-1$ then $n^2 - k - 1 > n^2 - n$ or the number of non-zero roots exceeds $n^2 - n$. But in § 2 it was shown that the number of non-zero roots does not exceed $n^2 - n$, hence $k \not< n-1$, and we must have $k = n-1$.

Killing has shown that the number of independent invariants of the transformations of the adjoint of any group can not be less than the rank of the group (*Math. Ann.*, Vol. XXXI, p. 266). Thus the number of independent invariants of the adjoint of G' can not be less than $n-1$. Moreover, the number of invariants of the adjoint of any group G_r is the same as the nullity of the matrix $\sum_{i=1}^{r} \alpha_i E'_i$ which can not exceed the number of zero roots of the characteristic equation $|\omega - \sum_{i=1}^{r} \alpha_i E_i| = 0$. Since the rank of G' is $n-1$ it follows that the characteristic equation of G' has exactly $n-1$ roots zero and therefore the number of invariants can not exceed $n-1$ and must, therefore, be exactly $n-1$. Hence:

VI. *The rank of the special linear homogeneous group in n variables is $n-1$, and is the same as the number of independent invariants of the transformations of its adjoint group.*

From equations (44) it follows that the characteristic equation,

$$|\omega-\sum_{i=1}^{n^2}\alpha_i E_i|=\omega|\omega-\sum_{i=1}^{n_2-1}\alpha_i H_i|=0,$$

of the general linear homogeneous group, G, has the same number of independent coefficients as the characteristic equation $|\omega-\sum_{i=1}^{n^2-1}\alpha_i H_i|=0$ of G'. Thus the rank of G is equal to the rank of G' and is $n-1$. Also since the nullity of $\sum_{i=1}^{n^2-1}\alpha_i H_i$, a matrix of order n^2-1, is $n-1$ it is evident that the nullity of $\sum_{i=1}^{n^2}\alpha_i E_i=\begin{pmatrix}\sum_{i=1}^{n^2-1}\alpha_i H_i, & 0\\ 0, & 0\end{pmatrix}$, a matrix of order n^2, is exactly n. We have then:

VII. *The rank of the general linear homogeneous group in n variables is $n-1$ and the number of independent invariants of the transformations of its adjoint group is n.*

Let the l distinct roots of $|\rho-\sum_{i=1}^{r}\alpha_i A_i|=0$ be respectively of multiplicity $\mu_1, \mu_2, \ldots, \mu_l$. The equation $|\omega-\sum_{i=1}^{r}\alpha_i E_i|=0$ has then at least $\mu_1^2+\mu_2^2+\ldots+\mu_l^2$ zero roots. If this equation has as many as n^2-r+2 zero roots it follows from (49) that $\Psi_{r-1}(\alpha)=0$. Therefore, $\Psi_{r-1}(\alpha)=0$ if $\mu_1^2+\mu_2^2+\ldots+\mu_l^2 \geqq n^2-r+2$. But Cartan has shown that, when the coefficient $\Psi_{r-1}(\alpha)$ of the characteristic equation of G_r vanishes identically, every transformation of G_r is commutative with at least one other transformation of G_r (*Thèses*, p. 32). Whence we have the following theorem:

VIII. *Let G_r be a subgroup with r parameters of the general linear homogeneous group in n variables; and let the l distinct roots of the characteristic equation of the general infinitesimal transformation of G_r be of multiplicity $\mu_1, \mu_2, \ldots, \mu_l$. If $\mu_1^2+\mu_2^2+\ldots+\mu_l^2 \geqq n^2-r+2$, then every infinitesimal transformation of G_r is commutative with at least one other infinitesimal transformation of this group.*

The group

$$A_1=x_1\frac{\partial}{\partial x_1},\quad A_2=x_1\frac{\partial}{\partial x_2},\quad A_3=x_1\frac{\partial}{\partial x_3},\quad A_4=x_1\frac{\partial}{\partial x_4},\quad A_5=x_3\frac{\partial}{\partial x_2},$$

$$A_6=x_3\frac{\partial}{\partial x_4},\quad A_7=x_4\frac{\partial}{\partial x_2},\quad A_8=x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+x_4\frac{\partial}{\partial x_4},$$

illustrates the above theorem. The characteristic equation of the general infinitesimal transformation is $(\rho-\alpha_1)(\rho-\alpha_4)^3=0$. Thus $\mu_1^2+\mu_2^2=10=n^2-r+2$.

This group contains no exceptional transformation but $(A_1, A_8)=(A_2, A_7)$ $=(A_3, A_4)=(A_5, A_6)=0$. The group given on page 446 also illustrates this theorem. In this case we have $\mu_1^2=9>7=n^2-r+2$, and the group contains two exceptional transformations.

By theorem (v) the characteristic equation of a simple or semi-simple group with r parameters and of rank k has $r-k$ distinct non-zero roots. I have shown that the number of distinct non-zero roots of the characteristic equation of G_r can not exceed l^2-l, and that the rank of G_r can not exceed $l-1$ or exceed l'. Suppose G is a simple group of rank k, then from Killing's theorem and what has just been stated $k\leqq l-1$, $k\leqq l'$ and $r-k\leqq l^2-l$. Therefore, $r\leqq l^2-1$ and $r\leqq l^2-l+l'$. We have then the following theorem:

IX. *Let G_r be a subgroup with r parameters of the general linear homogeneous group; let l be the maximum number of distinct roots of the characteristic equation of the general infinitesimal transformation of G_r and let l' be the number of these roots which are independent. If either l^2-1 or l^2-l+l' is less than r then the group can be neither simple nor semi-simple.*

The example given on page 448 illustrates this theorem for $l^2-1=3<8=r$ and the group contains the invariant subgroup A_2, A_3, A_4, A_5, and by III is integrable.

Since $\Psi_r(\alpha)=|\sum_{i=1}^{r}\alpha_i E'_i|$ is identically zero, the nullity of the matrix $\sum_{i=1}^{r}\alpha_i E'_i$ is at least 1. For a given infinitesimal transformation $\sum_{i=1}^{r}\alpha_i A_i$ of G_r, let the nullity of $\sum_{i=1}^{r}\alpha_i E'_i$ be unity, and let the equation $|\omega-\sum_{i=1}^{r}\alpha_i E'_i|=0$ have two roots zero. Then, for the given infinitesimal transformation $\sum_{i=1}^{r}\alpha_i A_i$ of G_r, $\Psi_{r-1}(\alpha)$ is annulled, but not all the first minors of $\sum_{i=1}^{r}\alpha_i E'_i$, and hence by theorem (iv) all the invariants of the adjoint, homogeneous and of degree non-zero, are annulled by the given infinitesimal transformation. Let $\mu_1, \mu_2, \ldots, \mu_l$ be the multiplicities of the distinct roots of the characteristic equation of $\sum_{i=1}^{r}\alpha_i A_i$. From (49) it follows that $\Psi_{r-1}(\alpha)=0$ if $|\omega-\sum_{i=1}^{r}\alpha_i E_i|=0$ has at least n^2-r+2 zero roots, which will be the case if $\sum_{i=1}^{l}\mu_i^2\geqq n^2-r+2$ (cf. p. 448); and the nullity of $\sum_{i=1}^{r}\alpha_i E'_i$ will not exceed unity if the nullity of $\sum_{i=1}^{r}\alpha_i E_i$ does not exceed n^2-r+1.

From (13) it follows that the nullity of $\sum_{i=1}^{r} \alpha_i E_i$ is equal to the nullity of $\sum_{i=1}^{r} \alpha_i (M_i - N_i)$. We have then the following theorem:

X. *Let G_r be a subgroup with r parameters of the general linear homogeneous group in n variables. Let the $l \leqq n$ distinct roots of the characteristic equation of any given infinitesimal transformation $\sum_{i=1}^{r} \alpha_i A_i$ of G_r be respectively of multiplicity $\mu_1, \mu_2, \ldots, \mu_l$. If $\sum_{i=1}^{l} \mu_i^2 \geqq n^2 - r + 2$ and the nullity of $\sum_{i=1}^{r} \alpha_i (M_i - N_i)$* (see p. 437) *does not exceed $n^2 - r + 1$, the invariants of the adjoint, homogeneous and of degree non-zero, are then all annulled for this infinitesimal transformation.*

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